A family of distortion premium principles: largest claims and reinsurance

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A premium principle is a rule that assigns an adequate price, called premium, to a risk to be insured. Formally, given a risk $X$, a premium principle is a function that assigns a non-negative real number to each value of $X$.

In the literature there are different approaches to determine premium principles (see Sordo, Castaño-Martínez and Pigueiras, 2016, and references therein for an overview).

Distortion risk measures have many convenient properties: they are monotone, translation invariant, positively homogeneous and, when the distortion function is concave, subadditive and, therefore, coherent.
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Given the risk $X$, let a random sample $X_1, X_2, \ldots, X_n$ of independent claims with the same distribution $F$ as the risk $X$ and denote by $X_{1:n} \leq \ldots \leq X_{n:n}$ the corresponding order statistics.

The total claim amount of the $n - i$ largest claims, given by

$$S_{i,n}(X) = \sum_{j=i+1}^{n} X_{j:n}, \quad 0 \leq i \leq n - 1,$$

has received some attention in the actuarial literature (see Embrechts and Klüppelberg, 1993, and Aebi et al., 1994).

Several reinsurance treaties (such as LCR and ECOMOR) are based on the largest claims during a given period of time.
Introduction
A family of premium principles
A convergence result
Largest claims and stochastic orders
Application to the comparison of reinsurance treaties
Conclusions

Framework

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A family of premium principles

Definition

Given the risk $X$ with distribution function $F$, we quantify the right-tail risk of $X$ by using the expected average of the $n - i$ largest claims, given by

$$T_{i,n}(X) = \frac{1}{n-i} \sum_{j=i+1}^{n} E[X_{j:n}], \quad 0 \leq i \leq n-1, \quad n \geq 2.$$
A family of premium principles

Theorem

Let $X$ be a risk with tail function $\bar{F}$ and finite mean $\mu$. Then, for $1 \leq i \leq n - 1$ and $n \geq 2$,

$$T_{i,n}(X) = \int_0^1 \text{TVaR}_p(X) d\beta_{i,n-i+1}(p),$$

where

$$\text{TVaR}_p(X) = \frac{1}{1 - p} \int_p^1 F^{-1}(t) \, dt, \quad p \in (0, 1)$$

is the tail-value-at-risk at level $p$ of $X$ and $\beta_{i,j}(p)$ is the Pearson's incomplete beta function (Pearson, 1934) with parameters $(i, j)$. 

Pigueiras, Castaño-Martínez, Sordo

Risk measures based on largest claims
Representations of $T_{i,n}(X)$

Three different representations

- $T_{i,n}(X)$ is a weighted area under the curve defined by $\text{TVaR}_p(X)$.

- $T_{i,n}(X) = E[X] + \epsilon_{i,n}(X)$, where $\epsilon_{i,n}(X)$ is the risk loading of the premium.

- $T_{i,n}(X)$ is a distortion premium principle with concave distortion function given by

$$h_{i,n}(t) = 1 - c_{i,n} \int_t^1 \int_p^1 (1 - u)^{i-1} u^{n-i-1} dudp.$$ 

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  \[ h_{i,n}(t) = 1 - c_{i,n} \int_t^1 \int_p^1 (1 - u)^{i-1} u^{n-i-1} du dp. \]

Some computations

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Table 1: $T_{i,n}(X)$, $X \sim Pa(2, 2)$

This computations suggest that sequences of the form $\{ T_{i,n} \}$ converge to $\text{TVaR}_p(X)$ as $n \to \infty$ and $i_n/n \to p$, for $p \in (0, 1)$. 
A convergence result

**Theorem**

Let $X$ be a risk with continuous tail function $\bar{F}$ and finite mean $\mu$. Given $p \in (0, 1)$, we have

$$\lim_{n \to \infty} T_{\lfloor np \rfloor, n} = TVaR_p(X).$$

where $\lfloor np \rfloor$ denotes the greatest integer less than or equal to $np$. 

**Introduction**

A family of premium principles

A convergence result

Largest claims and stochastic orders

Application to the comparison of reinsurance treaties

Conclusions
In the literature, there are different orderings that formalize the idea that one risk is more dangerous than another. We focus on the excess-wealth order.

**Definition**

Let $X$ and $Y$ be two risks with distribution functions $F$ and $G$, respectively. Then, $X$ is said to be smaller than $Y$ in the excess-wealth order (denoted by $X \preceq_{ew} Y$) if

$$
\text{ES}_p(X) \leq \text{ES}_p(Y), \text{ for all } p \in (0, 1)
$$

where

$$
\text{ES}_p(X) = E \left[ (X - F^{-1}(p))_+ \right].
$$

is the expected shortfall at level $p$. 

Now we characterize the excess-wealth order in terms of differences of the form

\[ T_{j,n}(X) - T_{i,n}(X) = \frac{1}{n-j} \sum_{k=j+1}^{n} E[X_{k:n}] - \frac{1}{n-i} \sum_{k=i+1}^{n} E[X_{k:n}] \]

which is a measure of the variability of \( X \).

**Theorem**

Given two risks \( X \) and \( Y \), \( X \leq_{ew} Y \) if and only if

\[ T_{j,n}(X) - T_{i,n}(X) \leq T_{j,n}(Y) - T_{i,n}(Y), \quad 1 \leq i \leq j < n, \quad n \geq 2. \]
Consider two collective of risks, denoted by \((X, N)\) and \((Y, M)\), respectively.

The collective \((X, N)\) is defined as follows: for a given period of time, let \(X_1, X_2, \ldots\) be a sequence of successive independent claim sizes with the same distribution \(F\) as \(X\) and let \(N\) describe the number of claims, which is assumed to be independent of the claim sizes \(\{X_i, i \geq 1\}\). We order the claims in increasing size resulting in the ordered sequence \(X_{1:N} \leq \ldots \leq X_{N:N}\).

The collective \((Y, M)\) is defined and ordered similarly.

A reinsurance policy is a contract, according to which part of the risk of an insurance company (the ceding company) is transferred to another insurance company (the reinsurance company), in exchange for receiving a premium (see Albrecher et al., 2017).
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Several types of reinsurance contracts based on large claims have been studied in the literature, see Ammeter (1964), Thépaut (1950) and Kremer (1983) among others.

Under a LCR (largest claims reinsurance) contract, the reinsurer agrees to cover the largest \( r \) claims, where \( r \geq 1 \) is fixed. For a collective \((X, N)\), the reinsured amount is

\[
L_r(X, N) = \sum_{i=1}^{r} X_{N-i+1:N}.
\]

A variant of LCR contract is called ECOMOR (Excédent du Coût Moyen Relatif) which is defined in Thépaut (1950) by

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H_r (X, N) = \sum_{i=1}^{r} X_{N-i+1:N} - rX_{N-r:N}.
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LCR and ECOMOR treaties

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Another way to formalize the idea that one risk is more dangerous than another is through the stop-loss order.

**Definition**

Let $X$ and $Y$ be two risks with distribution functions $F$ and $G$, respectively. Then, $X$ is said to be smaller than $Y$ in the stop-loss order (denoted by $X \leq_{sl} Y$) if

$$\pi_X(t) \leq \pi_Y(t), \text{ for all } t > 0,$$

where

$$\pi_X(t) = E[(X - t)_+] = \int_t^\infty F(x) \, dx, \quad t > 0,$$

with $X_+ = \max \{X, 0\}$, is the stop-loss premium.
Usual stochastic order

Definition

Given two risks $X$ and $Y$ with distribution functions $F$ and $G$, respectively, $X$ is said to be smaller than $Y$ in the stochastic order (denoted by $X \leq_{st} Y$) if

$$\text{VaR}_p(X) \leq \text{VaR}_p(Y) \text{ for all } p \in (0, 1).$$

where

$$\text{VaR}_p(X) = F^{-1}(p) = \inf \{x : F(x) \geq p\}$$

is the value-at-risk of the risk $X$ at level $p$. 
Corollary

Consider two collectives \((X, N)\) and \((Y, M)\). If \(X \leq_{sl} Y\) and \(N \leq_{st} M\), then 
\[ E[L_r(X, N)] \leq E[L_r(Y, M)], \text{ for all integer } r \geq 1. \]

Corollary

Consider two collectives \((X, N)\) and \((Y, M)\). Assume that \(X\) or \(Y\) is DFR (decreasing failure rate). If \(X \leq_{ew} Y\) and \(N \leq_{st} M\), then 
\[ E[H_r(X, N)] \leq E[H_r(Y, M)], \text{ for all integer } r \geq 1. \]
Comparison of net LCR and ECOMOR premiums

Consider two collectives \((X_i, N_i), i = 1, 2\), where the claim-size random variables \(X_i\) follow Pareto distributions with parameters \(\alpha_i\) and \(\beta_i\), \(i = 1, 2\), respectively. Assume that the claim-frequency random variables \(N_i\) follow Poisson distributions with parameters \(\lambda_i\), \(i = 1, 2\), respectively.

Example (comparing LCR treaties)

\[
\begin{align*}
\alpha_1 &> 1, \alpha_2 > 1 \\
\beta_1 &\geq \beta_2 \\
\frac{\alpha_1(\alpha_2-1)}{\alpha_2(\alpha_1-1)} &\leq \frac{\beta_2}{\beta_1} \\
\lambda_1 &\leq \lambda_2
\end{align*}
\]

\[\implies E[L_r (X_1, N_1)] \leq E[L_r (X_2, N_2)], \ r \geq 1.\]

Example (comparing ECOMOR treaties)

\[
\begin{align*}
\alpha_1 &> \alpha_2 > 1 \\
\alpha_2\beta_1 &\leq \alpha_1\beta_2 \\
\frac{\alpha_1(\alpha_2-1)}{\alpha_2(\alpha_1-1)} &\leq \frac{\beta_2}{\beta_1} \\
\lambda_1 &\leq \lambda_2
\end{align*}
\]

\[\implies E[H_r (X_1, N_1)] \leq E[H_r (X_2, N_2)], \ r \geq 1.\]
Conclusions

- Given a risk $X$, we have proved that the expected average of the $n - i$ largest claims drawn from a sample on $n$ independent copies of $X$ is a distortion risk measure with concave distortion function.

- We provided a convergence result that interprets $\text{TVaR}_p(X)$ as arithmetic average of the $100(1 - p)\%$ largest claims of $n$ independent claims, for $n$ large enough, with the same distribution as $X$.

- We have characterized the excess-wealth order in terms of the premiums $T_{i,n}$.

- We provided sufficient conditions for the comparison of ECOMOR and LCR reinsurance treaties in terms of the stop-loss order and the excess-wealth order of the claim-size distributions when the number of claims distributions are stochastically ordered.
References


References


