On a family of risk measures based on largest claims

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**A B S T R A C T**

Given a set of \( n \geq 2 \) independent and identically distributed claims, the expected average of the \( n - i \) largest claims, with \( 0 \leq i \leq n - 1 \), is shown to be a distortion risk measure with concave distortion function that can be represented in terms of mixtures of tail value-at-risks with beta mixing distributions. This result allows to interpret the tail value-at-risk in terms of the largest claims of a portfolio of independent claims. As an application, we provide sufficient conditions for stochastic comparisons of premiums in the context of large claims reinsurance.

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1. Introduction

Consider a random sample \( X_1, X_2, \ldots, X_n \) of independent claims with the same distribution \( F \) as \( X \) and denote by \( X_{1:n} \leq \cdots \leq X_{n:n} \) the corresponding order statistics. The total claim amount of the \( n - i \) largest claims, namely

\[
S_{n-i}(X) = \sum_{j=i+1}^{n} X_{j:n}, \quad 0 \leq i \leq n - 1,
\]

has received some attention in the actuarial literature, especially in the study of heavy-tailed distributions. For example, Embrechts and Klüppelberg (1993) and Aebi et al. (1994) study a measure of dangerousness of a claim distribution \( F \) by using the proportion of \( S_{n-i}(X) \) to the aggregate claim amount in the portfolio. In the context of collective risk models, several reinsurance treaties (such as LCR and ECOMOR) are based on the largest claims during a given period of time (see Section 2.5 and Chapter 10 in Albrecher et al., 2017, for an overview).

In this paper, given a random sample of \( n \) independent claims with the same distribution \( F \) as \( X \), we quantify the right-tail risk of \( X \) by using the expected average of the \( n - i \) largest claims, given by

\[
T_{i,n}(X) = \frac{1}{n-i} \sum_{j=i+1}^{n} E[X_{j:n}], \quad 0 \leq i \leq n - 1, \quad n \geq 2.
\]

The main result of this paper, stated in Section 2, shows that \( T_{i,n}(X) \) is a concave distortion risk measure, which means (see Yaari, 1987, and Section 1.1 in Goovaerts et al., 2010, for an overview) that there exists an increasing concave function \( h_{i,n} : [0, 1] \rightarrow [0, 1] \) with \( h_{i,n}(0) = 0 \) and \( h_{i,n}(1) = 1 \) (called distortion function) such that

\[
T_{i,n}(X) = \int_{0}^{\infty} h_{i,n}(\overline{F}(x)) \, dx,
\]

where \( \overline{F} = 1 - F \) is the tail function of \( X \). Distortion risk measures have many convenient properties: they are monotone, translation invariant, positively homogeneous and, when the distortion function is concave, subadditive and, therefore, coherent (see Section 2.6 in Denuit et al., 2005). In order to prove the main result, we first rewrite \( T_{i,n}(X) \) in terms of mixtures of tail value-at-risks with beta mixing distributions, thus obtaining an alternative representation for the premium principles defined in Sordo et al. (2016). In Section 3, as a consequence, we obtain a convergence result that interprets the tail value-at-risk of \( X \), which is defined by

\[
\text{TVaR}_p(X) = \frac{1}{1-p} \int_{p}^{1} F^{-1}(t) \, dt, \quad p \in (0, 1),
\]

where \( F^{-1}(t) = \inf\{x : F(x) \geq t\}, \quad t \in (0, 1) \), in terms of the largest claims of a portfolio of independent claims with the same distribution.

We consider in this paper two stochastic orders based on the notion of stop-loss premium: the stop-loss order and the excess-wealth order. For a risk \( X \) with deductible \( \ell \), the stop-loss

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premium is given by
\[ \pi_X(t) = E\left[(X - t)_+\right] = \int_t^\infty F(x) \, dx, \quad t > 0, \]
where \( X_+ = \max\{X, 0\} \). A stop-loss premium with retention \( t = F^{-1}(p) \) is called the expected shortfall at level \( p \), denoted by \( \text{ES}_p(X) \). Specifically, \( \text{ES}_p(X) = E\left[(X - F^{-1}(p))_+\right] \).

**Definition 1.** Let \( X \) and \( Y \) be two risks with distribution functions \( F \) and \( G \), respectively. Then, \( X \) is said to be smaller than \( Y \):

1. in the stop-loss order (denoted by \( X \leq_{sl} Y \)) if \( \pi_X(t) \leq \pi_Y(t) \), for all \( t > 0 \),
2. in the excess-wealth order (denoted by \( X \leq_{ew} Y \)) if \( \text{ES}_p(X) \leq \text{ES}_p(Y) \), for all \( p \in (0, 1) \).

In both orders, the smaller risk is the less risky: while \( X \) being smaller than \( Y \) in the stop-loss order means that \( X \) is smaller in size and less variable than \( Y \), in the excess-wealth order means that \( X \) is simply less variable than \( Y \). Much of the theory on these orders can be found in the book Shaked and Shanthikumar (2007). For actuarial applications of the stop-loss order, see Denuit et al. (2005) and Müller and Stoyan (2002) and for the excess-wealth order, see Sordo et al. (2018) and references therein.

**Lea de Caí and Cárcamo (2006) showed that**

\[ X \leq_{sl} Y \quad \text{if and only if} \quad T_{i,n}(X) \leq T_{j,n}(Y), \quad 1 \leq i < n, \quad n \geq 2. \]

Motivated by this characterization of the stop-loss order, we consider, in Section 4, variability measures of the form \( T_{i,n}(X) - T_{j,n}(X) \) with \( 1 \leq i < j \leq n - 1 \) and \( n \geq 2 \), to characterize the excess wealth order. In Section 5 we give applications in the context of two classical reinsurance treaties based on large claims: the LCR (largest claim reinsurance) treaty proposed by Ammeter (1964) and the ECOMOR (Excédent du Coût Moyen Relatif) treaty introduced by Thépaut (1950).

Throughout this paper, we consider non-negative random variables (or risks) with continuous and strictly increasing distribution functions.

### 2. A family of risk measures

Let \( X_1, \ldots, X_n \) be independent and identically distributed claims with distribution function \( F \) and finite mean \( \mu \). Consider \( X_1 \leq X_2 \leq \cdots \leq X_n \), the corresponding order statistics. The distribution function of \( X_{in} \) (\( i = 1, \ldots, n \)) is

\[ F_{in}(x) = \beta_{n-i+1}(F(x)), \quad x \geq 0, \]

where

\[ (i,j)(p) = \int_{0}^{p} \frac{(i+j-1)!}{(i-1)! (j-1)!} (1-t)^{i-1} (1-t)^{j-1} \, dt, \quad 0 \leq p \leq 1, \]

is the Pearson’s incomplete beta function (see Pearson, 1934) with parameters \((i,j)\). If \( F \) is absolutely continuous with density function \( f \), the density function of \( X_{in} \) (\( i = 1, \ldots, n \)) is

\[ f_{in}(x) = \frac{n!}{(i-1)! (n-i)!} \frac{F(x)^{i-1}}{[F(x)]^{n-i}} f(x), \quad x \geq 0. \]

Before stating the main result of this section, we require the following recurrence relations for order statistics.

**Lemma 2.** We have:

(a) \[ \sum_{j=i}^{n} F_{jn}(x) = n F(x) F_{in-1}(x) + (n-i) F_{in}(x), \quad 1 \leq i \leq n-1, \quad n \geq 2, \]

(b) \[ \frac{n}{n-i} f_{in-1}(x) F(x) = f_{in}(x), \quad 1 \leq i \leq n-1, \quad n \geq 2, \]

**Proof.** We only prove (a) (part (b) is easy). Let \( 1 \leq i < n, \quad n \geq 2 \). Since

\[ F_{in}(x) = \sum_{k=0}^{n-i} \binom{n}{k} (F(x))^{k} (\bar{F}(x))^{n-k} \]

we have

\[ \sum_{j=i+1}^{n} F_{jn}(x) = \sum_{j=i+1}^{n} \sum_{k=0}^{n-j} \binom{n}{k} (F(x))^{k} (\bar{F}(x))^{n-k} \]

\[ = (n-i) \sum_{k=0}^{n-i} \binom{n}{k} (F(x))^{k} (\bar{F}(x))^{n-k} \]

\[ + \sum_{j=i}^{n-i} \sum_{k=0}^{n-j} \binom{n}{k} (F(x))^{k} (\bar{F}(x))^{n-k} \]

\[ = (n-i) \bar{F}_{in}(x) + \sum_{k=0}^{n-j} \binom{n}{k} (n-k)(F(x))^{k} (\bar{F}(x))^{n-k} \]

\[ = (n-i) \bar{F}_{in}(x) + n \bar{F}_{in}(x) - \bar{F}_{in}(x) \]

which concludes the proof.

Next we prove that the risk measure \( T_{i,n}(X) \), defined in (1), can be represented by mixtures of tail value-at-risks, with beta mixing distributions.

**Theorem 3.** Let \( X \) be a risk with tail function \( \bar{F} \) and finite mean \( \mu \). Then, for \( 1 \leq i \leq n - 1 \) and \( n \geq 2 \),

\[ T_{i,n}(X) = \int_{0}^{1} TVaR_{p}(X) \, dp, \quad \bar{T}_{i,n}(X) = \int_{0}^{1} TVaR_{p}(X) \, dp. \]

**Proof.** From (1) we have

\[ T_{i,n}(X) = \frac{1}{n-i} \int_{0}^{\infty} \sum_{j=i+1}^{n} n F_{jn}(x) dx \]

\[ = \frac{n}{n-i} \int_{0}^{\infty} \bar{F}(x) F_{in-1}(x) dx + \int_{0}^{\infty} \bar{F}_{in}(x) dx \]

\[ = \frac{n}{n-i} \int_{0}^{\infty} \bar{F}(x) \left( \int_{0}^{\infty} f_{in-1}(y) \, dy \right) dx \]

\[ + \int_{0}^{\infty} \left( \int_{0}^{\infty} f_{in}(y) \, dy \right) dx \]

\[ = \int_{0}^{\infty} \bar{F}(x) \left( \int_{0}^{\infty} f_{in}(y) \, dy \right) dx + \int_{0}^{\infty} \left( \int_{0}^{\infty} f_{in}(y) \, dy \right) dx \]

\[ = \int_{0}^{\infty} \left( P(X > x | X > y) f_{in}(y) \right) dy dx \]

\[ = \int_{0}^{\infty} TVaR_{p}(X) \, dp, \]

where we have used Lemma 2(a) in (*) and Lemma 2(b) in (**).

Consequently, for \( 1 \leq i \leq n - 1 \) and \( n \geq 2 \), \( T_{i,n}(X) \) coincides with the premium principle studied in Sordo et al. (2016). By putting together Theorem 3 and Theorem 10 in Sordo et al. (2016), we deduce that \( T_{i,n}(X) \) is a distortion risk measure with concave distortion function.

**Corollary 4.** Let \( X \) be a risk with tail function \( \bar{F} \) and finite mean \( \mu \). Then, for \( 1 \leq i \leq n - 1 \) and \( n \geq 2 \), \( T_{i,n}(X) \) is a distortion risk.
measure with concave distortion function
\[ h_{\alpha}(t) = 1 - c_{\alpha} \int_{t}^{1} (1 - u)^{-1/\alpha} - u^{-1/\alpha} du, \]
where \( c_{\alpha} = \frac{n!}{(1 - \alpha)(n - 1)!} \).

3. A convergence result

Tables 1 to 3 in Sordo et al. (2016) suggest that sequences of the form \( T_{n,\alpha} \) converge to TVaR\(_{p}(X)\) as \( n \to \infty \) and \( \alpha_n / n \to \rho \), for \( \rho \in (0,1) \). We formalize the corresponding convergence result in terms of the sequence \( T_{\{\alpha\},n} \) for \( \rho \in (0,1) \), where \( \{\alpha\} \) denotes the greatest integer less than or equal to \( \alpha \).

The following result interprets TVaR\(_{p}(X)\) as arithmetic average of the \((100(1 - p))\) percent largest claims of a portfolio of \( n \) independent claims with the same distribution function \( F \).

**Theorem 5.** Let \( X \) be a risk with continuous tail function \( T \) and finite mean \( \mu \). Given \( p \in (0,1) \), we have
\[
\lim_{n \to \infty} T_{\{\alpha\},n} = TVaR_{p}(X),
\]
which proves the result. ■

Another motivation to study the limit of \( T_{\{\alpha\},n} \) is the following. Given \( n \), one possibility for selecting \( i \) in (1) is to require that the \( n - i \) largest claims consume a fixed proportion of the total claim volume. This problem has already been addressed in the actuarial literature (see Section 8.2 in Embrechts et al., 1997). For \( p \in (0,1) \), the random variable
\[
R_{n}(p) = \frac{X_{[\{\alpha\},n]} + \cdots + X_{n}}{S_{n}},
\]
where \( S_{n} = \sum_{j=1}^{n} X_{j} \), represents the proportion of the \( \{\alpha\} \) largest claims to the aggregate claim amount \( S_{n} \). Aebi et al. (1994) prove that, as \( n \to \infty \),
\[
\sup_{p \in [0,1]} |R_{n}(p) - L(p)| \to 0 \text{ a.s.}
\]
where
\[
L(p) = \frac{1}{\mu} \int_{1-p}^{1} F^{-1}(t) dt, \quad p \in (0,1]
\]
is the “dual” Lorenz curve used in economics to measure income inequality. Embrechts et al. (1997, p. 433), think of (6) as a large claim index that measures the extent to which the 100p percent largest claims in a portfolio contribute to the total claim amount.

In a similar fashion, by taking expectations in both the numerator and denominator of (5), we consider the index
\[
h_{\alpha}(p) = \frac{E[X_{\{\alpha\},n} + \cdots + X_{n}]}{E[S_{n}]} = \frac{[\alpha\]}{n\mu} T_{\{\alpha\},n}.
\]
It follows from Theorem 5 that
\[
h_{\alpha}(p) \to L(p) \quad \text{as} \quad n \to \infty, \quad p \in (0,1),
\]
which offers a similar interpretation for \( L(p) \) from a different perspective.

4. Largest claims and the excess-wealth order

Let \( 1 \leq i < j \leq n - 1 \). The difference between the expected average risk of the \( n - j \) largest claims and the \( n - i \) largest claims, given by
\[
T_{j,n}(X) - T_{i,n}(X) = \frac{1}{n-j} \sum_{k=j+1}^{n} E[X_{k:n}] - \frac{1}{n-i} \sum_{k=i+1}^{n} E[X_{k:n}]\quad (7)
\]
is a measure of the variability of \( X \). For example,
\[
T_{n-1,n}(X) - T_{n-2,n}(X) = \frac{1}{2} E[X_{n:n} - X_{n-1:n}]
\]
is \(1/2\) of the last sample spacing (difference of order statistics), a measure of interest in different contexts (see, for example, Li, 2005 and Kocher et al., 2007). Some recent papers (Belzunce et al., 2016b; Schweizer and Szech, 2017; Sordo and Psarrakos, 2017) have obtained sufficient conditions, in terms of the excess-wealth order, for the comparison of spacings under different assumptions. The aim of this section is to characterize the excess-wealth order in terms of differences of the form (7). Before stating the result, we recall that \( X \) and \( Y \), with distribution functions \( F \) and \( G \) respectively, are ordered in the usual stochastic order, denoted by \( X \leq_{st} Y \), if \( E[F(x)] \leq E[G(y)] \) for all increasing functions \( f \). Recall also from (3.3) and (3.4) in Shaked and Shanthikumar (2007) that \( X \leq_{ew} Y \) if and only if the function
\[
\psi(p) = \frac{1}{1-p} \int_{p}^{1} (G^{-1}(t) - F^{-1}(t)) dt, \quad p \in (0,1),
\]
is increasing or, equivalently, if
\[
\frac{1}{1-p} \int_{p}^{1} (F^{-1}(t) - G^{-1}(t)) dt \leq F^{-1}(p) - G^{-1}(p), \quad p \in (0,1).
\]

**Theorem 6.** Given two risks \( X \) and \( Y \), \( X \leq_{ew} Y \) if and only if
\[
T_{j,n}(X) - T_{i,n}(X) \leq T_{j,n}(Y) - T_{i,n}(Y), \quad 1 \leq i \leq j \leq n - 1, \quad n \geq 2.
\]

**Proof.** Assume \( X \leq_{ew} Y \). Using (3), for \( 1 \leq i \leq j \leq n - 1 \) and \( n \geq 2 \), we can write
\[
T_{i,n}(Y) - T_{i,n}(X) = \int_{1-p}^{1} (TVaR_{p}(Y) - TVaR_{p}(X)) d\beta_{i,n-i+1}(p)\quad (10)
\]
\[
= \int_{1-p}^{1} \psi(p) d\beta_{i,n-i+1}(p)
\]
\[= E[\psi(U_{n})],\]
where the function \( \psi \), given by (8), is increasing and \( U_{n} \) denotes the \( i \)th order statistics from a standard uniform distribution (which follows a beta distribution of parameters \( i \) and \( n - i + 1 \)). Analogously \( T_{j,n}(Y) - T_{i,n}(X) = E[\psi(U_{n})] \), where \( U_{n} \) is the \( j \)th order statistics from a standard uniform distribution. Since \( U_{n} \leq_{st} U_{j} \) for \( 1 \leq i \leq j \leq n - 1 \), we have
\[E[\psi(U_{n})] \leq E[\psi(U_{j})], \quad 1 \leq i \leq j \leq n - 1, \quad n \geq 2,
\]
which, using (10), is the same as (9).

To prove the converse, let \( 0 < p < q < 1 \). For \( n \) given, we choose in (9) the integers \( j = \lfloor nq \rfloor \) and \( i = \lfloor np \rfloor \). Taking limits as \( n \to \infty \) and using (4) it follows that
\[TVaR_{p}(X) - TVaR_{q}(X) \leq TVaR_{p}(Y) - TVaR_{q}(Y), \quad 0 < p < q < 1.
\]
This is equivalent to say that (8) is increasing, which means \( X \leq_{ew} Y \). ■
5. Application to the comparison of ECOMOR and LCR treaties

A reinsurance policy is a contract, according to which part of the risk of an insurance company (the ceding company) is transferred to another insurance company (the reinsurer company), in exchange for receiving a premium (see Albrecher et al., 2017). Several types of reinsurance contracts based on large claims have been studied in the literature. We focus on LCR and ECOMOR contracts.

Let \( (X, N) \) be a collective of risks defined as follows. For a given period of time, let \( X_1, X_2, \ldots \) be a sequence of successive independent claim sizes with the same distribution \( F \) as \( X \) and let \( N \) describe the number of claims, which is assumed to be independent of the claim sizes \( \{X_i, i \geq 1\} \). We order the claims in increasing size resulting in the ordered sequence \( X_{1:N} \leq \cdots \leq X_{N:N} \). Under a LCR (largest claims reinsurer) contract, the reinsurer agrees to cover the largest \( r \) claims, where \( r \geq 1 \) is fixed. The reinsured amount is

\[
L_r(X, N) = \sum_{i=1}^{r} X_{N-i+1:N}. 
\]


Given two collectives with the same number of claims distributions, the next theorem provides a sufficient condition for the stop-loss order of the respective LCR amounts. In order to prove it, we require the notion of copula and some concepts of positive dependence (see Joe, 1997, for details). A copula \( C \) is a multivariate distribution function with uniform marginals on [0, 1]. If \( H \) is a \( n \)-dimensional distribution function with marginal distribution functions \( F_1, \ldots, F_n \), then there exists a \( n \)-copula \( C \) such that, for all \( (x_1, \ldots, x_n) \in \mathbb{R}^n \), we have \( H(x_1, \ldots, x_n) = C(F_1(x_1), \ldots, F_n(x_n)) \). Moreover, if \( F_1, \ldots, F_n \) are continuous, then \( C \) is unique.

**Definition 7.** A random vector \( (X_1, \ldots, X_n) \) with joint density function \( f \) is multivariate totally positive of order 2 (MTP2) if

\[
f(x) f(y) \leq f(x \vee y) f(x \wedge y),
\]

where \( \vee \) and \( \wedge \) are the corresponding lattice operations, i.e.

\[
x \vee y = (x_1 \vee y_1, \ldots, x_n \vee y_n),
\]

and

\[
x \wedge y = (x_1 \wedge y_1, \ldots, x_n \wedge y_n).
\]

**Definition 8.** A random vector \( (X_1, \ldots, X_n) \) is conditionally increasing in sequence (CIS) if, for \( i = 2, \ldots, n \),

\[
\{X_i|X_1 = x_1, \ldots, X_{i-1} = x_{i-1}\} \leq_{\mathrm{st}} \{X_i|X_1 = x_1', \ldots, X_{i-1} = x_{i-1}'\}
\]

whenever \( x_j \leq x_j', j = 1, 2, \ldots, i-1 \).

**Definition 9.** A random vector \( (X_1, \ldots, X_n) \) is conditionally increasing (CI) if, and only if, the random vector

\[
X_{\pi} = (X_{\pi(1)}, \ldots, X_{\pi(n)})
\]

is CIS for all permutations \( \pi \in \Pi_n \).

For properties and applications of MTP2 and CI, see Karlin and Rinott (1980). In particular, it is well-known (Müller and Scarsini, 2001) that MTP2 implies CI.

Now we can prove the following result.

**Theorem 10.** Let \( (X, N) \) and \( (Y, N) \) be two collectives of risks. If \( X_{n:n} \leq \mathrm{st} Y_{n:n} \) for \( n \geq 1 \), then

\[
\sum_{i=1}^{r} X_{N-i+1:N} \leq \mathrm{st} \sum_{i=1}^{r} Y_{N-i+1:N}
\]

for all integers \( r \geq 1 \).

**Proof.** On the one hand, it follows from Theorem 3.11 in Balakrishnan et al. (2012) that \( X_{n:n} \leq \mathrm{st} Y_{n:n} \) implies

\[
X_{n:n} \leq \mathrm{st} Y_{n:n}, \quad i = 1, \ldots, n.
\]

On the other hand, it is well-known (see Navarro and Spizzichino, 2010) that the connecting copula of the order statistics of \( n \) independent and identically distributed random variables \( X_1, \ldots, X_n \) with the same marginal distribution function \( F \) does not depend on \( F \). Moreover, a random vector of order statistics is MTP2 (see Prop. 3.11 in Karlin and Rinott, 1980) and is therefore conditionally increasing. Since marginalization preserves the copula and the property MTP2 (see Karlin and Rinott, 1980, p. 471) the vectors \( (X_{n-r+1:n}, \ldots, X_{n:n}) \) and \( (Y_{n-r+1:n}, \ldots, Y_{n:n}) \) are also conditionally increasing and have the same copula. From this fact, (12) and Corollary 2.7 in Balakrishnan et al. (2012) it follows that

\[
\sum_{i=1}^{r} X_{N-i+1:N} \leq \mathrm{st} \sum_{i=1}^{r} Y_{N-i+1:N}.
\]

Now, the desired result follows from Theorem 4.A.8(b) in Shaked and Shanthikumar (2007).

Balakrishnan et al. (2012, Example 4.16) show that, under some conditions on the parameters, Weibull random variables satisfy the assumptions of Theorem 10.

Note that, in general, the stop-loss order is not preserved under the formation of order statistics and, therefore, the condition \( X \leq \mathrm{st} Y \) is not sufficient to ensure (12). However, the latter condition is sufficient to order the LCR net premiums of two collectives \( (X, N) \) and \( (Y, M) \), whenever \( N \) is stochastically smaller than \( M \).

**Corollary 11.** Consider two collectives \( (X, N) \) and \( (Y, M) \). If \( X \leq \mathrm{st} Y \) and \( N \leq \mathrm{st} M \), then \( E[L_r(X, N)] \leq E[L_r(Y, M)] \) for all integers \( r \geq 1 \).

**Proof.** Let \( r \geq 1 \) be a fixed integer. The net risk premium for the LCR treaty is given by

\[
E[L_r(X, N)] = E[E[L_r(X, N)] | N]]
\]

\[
= \sum_{n=r}^{\infty} E \left[ \sum_{i=1}^{r} X_{n-i+1:n} \right] \Pr[N = n]
\]

\[
= \sum_{n=r}^{\infty} \sum_{i=1}^{r} \Pr[N = n] \Pr[X_{n-i+1:n}]
\]

Now we prove that \( T_{n-r,n}(X) \) is an increasing function of \( n \). In order to see it, recall from Theorem 1.C.37 in Shaked and Shanthikumar (2007) that \( E[X_{\alpha}] \leq E[X_{\alpha}] \) whenever \( j \leq i \) and \( m - j \geq n - i \). In particular,

\[
E[X_{m+1:n}] \leq E[X_{m+1:n}] \] for \( j = 1, \ldots, n \).

Therefore

\[
rT_{n-r,n}(X) = \sum_{j=1}^{r} E[X_{n-j+1:n}] \]

\[
\leq \sum_{j=1}^{r} E[X_{n-j+1:n}] = rT_{n+1-r,n}(X).
\]
Then, 
\[
E[1_r(X, N)] = r \sum_{n=r}^{\infty} T_{n-r,n}(X) \Pr[N = n]
\]
\[
\leq r \sum_{n=r}^{\infty} T_{n-r,n}(X) \Pr[M = n]
\]
\[
\leq r \sum_{n=r}^{\infty} T_{n-r,n}(Y) \Pr[M = n]
\]
\[
= E[1_r(Y, M)],
\]
where the first inequality follows from the facts that \( N \leq_{st} M \) and the function
\[
\beta_r(n) = \begin{cases} 
0, & n < r \\
T_{n-r,n}(X), & n \geq r
\end{cases}
\]
is increasing in \( n \) and the second inequality follows from (2).

A variant of LCR contract is called ECOMOR (Excédent du Coût Moyen Relatif). It is defined in Thépaut (1950) by
\[
H_r(X, N) = \sum_{i=1}^{r} X_{n-i+1:n} - rX_{n-r:n}.
\]
We focus on the net premium of ECOMOR in the case of DFR claim size distributions (which are typically heavy-tailed). Recall that a risk \( X \) is said to be DFR (or to have decreasing failure rate) if its tail function \( T \) is log-convex. Examples of DFR claim size distributions include gamma and Weibull distributions with shape parameters less than one and Pareto. 

For two collectives \((X, N)\) and \((Y, M)\) such that either \( X \) or \( Y \) is DFR, the following result gives a sufficient condition for ordering the corresponding net premiums under the same ECOMOR treaty.

**Corollary 12.** Consider two collectives \((X, N)\) and \((Y, M)\). Assume that \( X \) or \( Y \) is DFR. If \( X \leq_{ev} Y \) and \( N \leq_{st} M \), then \( E[H_r(X, N)] \leq E[H_r(Y, M)] \) for all integers \( r \geq 1 \).

**Proof.** Let \( r \geq 1 \) be a fixed integer and assume that \( X \) is DFR (if \( Y \) is DFR, the proof is similar). Applying the same reasoning as in the proof of Corollary 11, we have
\[
E[H_r(X, N)] = r \sum_{n=r}^{\infty} (T_{n-r,n}(X) - E[X_{n-r:n}]) \Pr[N = n]
\]
\[
= (r + 1) \sum_{n=r}^{\infty} (T_{n-r,n}(X) - T_{n-r-1,n}(X)) \Pr[N = n],
\]
where we define \( T_{n-r,n}(X) = \frac{1}{r} \sum_{j=1}^{r} E[X_{n-j+1:n} - X_{n-r:n}] \), which, using that \( X \) is DFR, is an increasing function of \( n \) (see Theorem 4.1 in Hu and Wei, 2001). Therefore
\[
E[H_r(X, N)] = (r + 1) \sum_{n=r}^{\infty} (T_{n-r,n}(X) - T_{n-r-1,n}(X)) \Pr[N = n]
\]
\[
\leq (r + 1) \sum_{n=r}^{\infty} (T_{n-r,n}(X) - T_{n-r-1,n}(X)) \Pr[M = n]
\]
\[
\leq (r + 1) \sum_{n=r}^{\infty} (T_{n-r,n}(Y) - T_{n-r-1,n}(Y)) \Pr[M = n]
\]
\[
= E[H_r(Y, M)],
\]
where the first inequality follows from the facts that \( N \leq_{st} M \) and the function
\[
\beta_r(n) = \begin{cases} 
0, & n < r \\
T_{n-r,n}(X) - T_{n-r-1,n}(X), & n \geq r
\end{cases}
\]
is increasing in \( n \) and the second inequality follows from Theorem 6.

**Example 13.** To illustrate Corollaries 11 and 12, we consider two collectives \((X_i, N_i), i = 1, 2\), where the claim-size random variables \( X_i \) follow Pareto distributions with parameters \( \alpha_i \) and \( \beta_i \), \( i = 1, 2 \), respectively. Recall that \( X \) follows a Pareto distribution with parameters \( \alpha \) and \( \beta \), denoted by \( X \sim \text{Pareto} (\alpha, \beta) \), if its tail function is given by
\[
\bar{F}(x) = \left( \frac{\beta}{x + \beta} \right)^\alpha, x > 0.
\]
Assume that the claim-frequency random variables \( N_i \) follow Poisson distributions with parameters \( \lambda_i \), \( i = 1, 2 \), respectively. Taking into account the ordering conditions for Pareto and Poisson distributions (see tables 2.1, 2.2 and 2.5 in Belzunce et al., 2016a) it follows from Corollary 11 that
\[
\alpha_1 > \alpha_2 > 1 \quad \alpha_2 \beta_1 \beta_2 \leq \alpha_1 \beta_1 \beta_2 \quad \Rightarrow E[L_r(X_1, N_1)] \leq E[L_r(X_2, N_2)], \quad r \geq 1.
\]
Moreover, since Pareto distributions are DFR, it follows from Corollary 12 that
\[
\alpha_1 > \alpha_2 > 1 \quad \alpha_2 \beta_1 \beta_2 \leq \alpha_1 \beta_1 \beta_2 \quad \Rightarrow E[H_r(X_1, N_1)] \leq E[H_r(X_2, N_2)], \quad r \geq 1.
\]

6. Conclusions

Given a risk \( X \), we have proved that the expected average of the \( n - i \) largest claims drawn from a sample on \( n \) independent copies of \( X \) is a distortion risk measure with concave distortion function. We have also provided sufficient conditions for the comparison of two reinsurance treaties based on largest claims (ECOMOR and LCR) in terms of the stop-loss order and the excess-wealth order of the claim-size distributions when the number of claims distributions are stochastically ordered.

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**References**


