A BSDE Approach to Derivatives Valuation with Regime Switching

Weijie Pang, McMaster University
Stephan Sturm, Worcester Polytechnic Institute

Oline International Conference in Actuarial Science, Data Science and Finance

April 29, 2020

Motivation

- Most current literatures do not take freezing of repurchase (Repo) market during the financial crisis into account.
- We introduce a process to describe the switching between normal financial status and financial crisis.
- We pricing a European option under funding spread and market illiquidity?

Repurchase Agreement Market

Definition

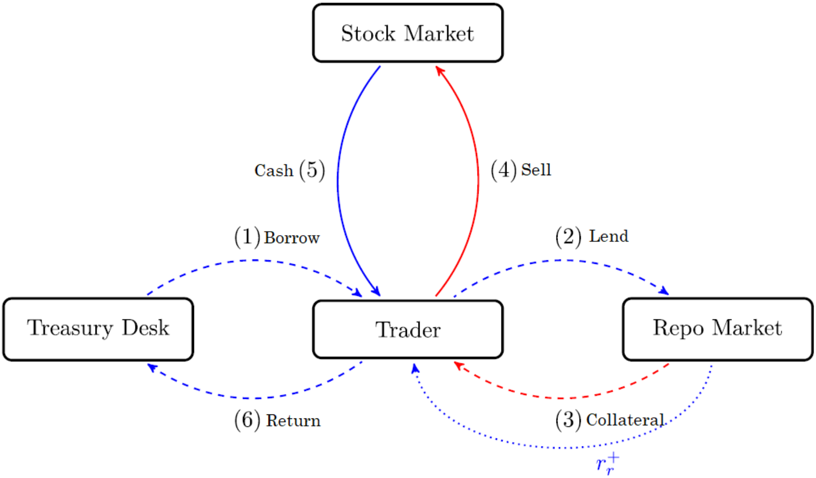
The sale and repurchase agreement (Repo Market) A sale of a security combined with an agreement to repurchase the same security at a specified price at the end of the contract.

- **Repo rate:**

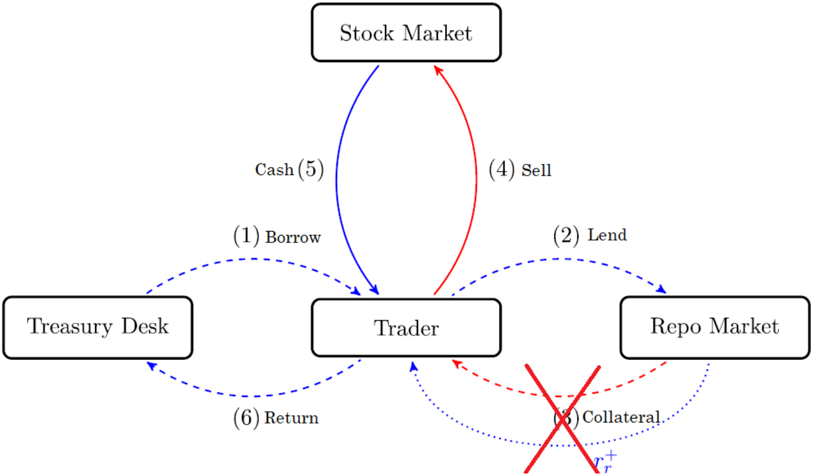
$$\frac{\text{The maturity price of the security - The inital price of the security}}{\text{The initial price of the security}}$$

- **Collateral:**
  - General Collaterals
  - Specific Collaterals

Repurchase Agreement Market

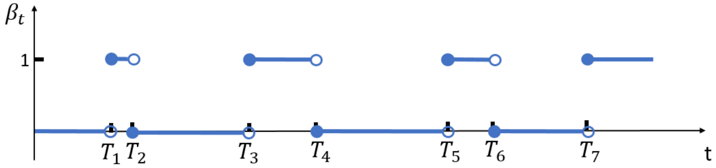


Repurchase Agreement Market



Alternating Renewal Processes β_t

Figure: One path of the process β_t



- $\beta_t = \sum_{i=1}^{\infty}(-1)^{i+1}\mathbb{1}_{\{T_i \leq t\}}$ is a stochastic process without independent increment property.
- T_i : the i^{th} alternating time ($T_0 = 0$)
- $U_i = T_{2i-1} - T_{2i-2} \sim \exp(\lambda_U)$: i^{th} financial normal status
- $V_i = T_{2i} - T_{2i-1} \sim \exp(\lambda_V)$: i^{th} financial crisis status

Financial Accounts

Under a valuation measure Q with discount interest rate r_D,

- Stock account: $dS_t = r_D S_t dt + \sigma S_t dW_t^Q$
- Repo account: $dB_t^{r_r^\pm} = r_r^\pm B_t^{r_r^\pm} dt$
- Funding account: $dB_t^{r_f^\pm} = r_f^\pm B_t^{r_f^\pm} dt$
- Collateral account: $dB_t^{r_c^\pm} = r_c^\pm B_t^{r_c^\pm} dt$.
- Risky bond account: $dP_t^I = r_D P_t^I dt - P_{t-}^I d\varpi_t^{I,Q}$,
  $dP_t^C = r_D P_t^C dt - P_{t-}^C d\varpi_t^{C,Q}$,

Performances During the Financial Crisis

- Repo account: $B_t^{r_r}(1-\beta_t)$
- Stock account: $(1-\beta_t \mathbb{1}_{\{\xi_t<0\}})\xi_t S_t$
- Assumptions:
  - In normal financial status: stock account is financed by the Repo account $(1-\beta_t)\psi_t^r B_t^{r_r} = -(1-\beta_t)\xi_t S_t$
  - Always:

$$
(1-\beta_t)\psi_t^r B_t^{r_r} + \beta_t \xi_t^f B_t^{r_f} =
$$
$$
\beta_t V_t + \beta_t \psi_t^c B_t^{r_c} - (1-\beta_t \mathbb{1}_{\{\xi_t<0\}})\xi_t S_t - \beta_t \xi_t^I P_t^I - \beta_t \xi_t^C P_t^C
$$

Hedging Portfolio

- Hedging portfolio is

$$
\begin{aligned}
V_t(\varphi) = (1 - \beta_t \mathbb{1}_{\{\xi_t<0\}})\xi_t S_t + \xi_t^I P_t^I + \xi_t^C P_t^C + \xi_t^f B_t^{r_f} \\
+ (1 - \beta_t)\psi_t^r B_t^{r_r} - \psi_t^c B_t^{r_c}
\end{aligned}
$$

- Dynamic of V_t is

$$
\begin{aligned}
dV_t =& \Big((1 - \beta_t \mathbb{1}_{\{\xi_t<0\}}) r_D \xi_t S_t + r_D \xi_t^I P_t^I + r_D \xi_t^C P_t^C + r_f \xi_t^f B_t^{r_f} \\
&+ (1 - \beta_t) r_r \psi_t^{r_r} B_t^{r_r} - r_c \psi_t^c B_t^{r_c} \Big) dt \\
&+ \sigma \xi_t (1 - \beta_t \mathbb{1}_{\{\xi_t<0\}}) S_t dW_t^{\mathbb{Q}} - \xi_t^I P_{t-}^I d\varpi_t^{I,\mathbb{Q}} - \xi_t^C P_{t-}^C d\varpi_t^{C,\mathbb{Q}}
\end{aligned}
$$

The Total Valuation Adjustment(XVA)

Definition

The Total Valuation Adjustment (XVA) is an adjustment made to the fair value of a derivatives contract to take funding and credit risk into account.

The Total Valuation Adjustment(XVA)

Definition

The Total Valuation Adjustment (XVA) is an adjustment made to the fair value of a derivatives contract to take funding and credit risk into account.

The seller's and buyer's XVA are defined as

$$\text{XVA}_t^+ := V_t^+ - \hat{V}_t, \quad \text{XVA}_t^- := V_t^- - \hat{V}_t.$$

Here,

$$d\hat{V}_t = r_D \hat{V}_t dt + \hat{Z}_t dW_t^{\mathbb{Q}},$$
$$\hat{V}_T = \Theta,$$

where Θ is the terminal value of the claim from the point of view the third party.

XVA BSDE

$$-d\mathrm{XVA}_t^{\pm} = \tilde{f}^{\pm}(t, \mathrm{XVA}_t^{\pm}, \tilde{Z}_t^{\pm}, \tilde{Z}_t^{I,\pm}, \tilde{Z}_t^{C,\pm}; \beta, \hat{V}, \hat{Z})dt$$
$$- \tilde{Z}_t^{\pm} dW_t^{\mathbb{Q}} - \tilde{Z}_t^{I,\pm} d\varpi_t^{I,\mathbb{Q}} - \tilde{Z}_t^{C,\pm} d\varpi_t^{C,\mathbb{Q}},$$
$$\mathrm{XVA}_\tau^{\pm} = \tilde{\theta}_I(\hat{V}_\tau)\mathbb{1}_{\{\tau^I < \tau^C \wedge T\}} + \tilde{\theta}_C(\hat{V}_\tau)\mathbb{1}_{\{\tau^C < \tau^I \wedge T\}},$$

where

$$Z_t = \sigma(1 - \beta_t \mathbb{1}_{\{\xi_t < 0\}})\xi_t S_t, \quad Z_t^I = -\xi_t^I P_{t-}^I,$$
$$Z_t^C = -\xi_t^C P_{t-}^C, \quad \tilde{Z}_t^{\pm} := Z_t^{\pm} - \hat{Z}_t,$$
$$\tilde{Z}_t^{I,\pm} := Z_t^{I,\pm}, \quad \tilde{Z}_t^{C,\pm} := Z_t^{C,\pm},$$
$$\tilde{\theta}_I(\hat{v}) := -L_I((1-\alpha)\hat{v})^+,$$
$$\tilde{\theta}_C(\hat{v}) := L_C((1-\alpha)\hat{v})^-.$$

XVA BSDE

The generator functions $\tilde{f}^+$ and $\tilde{f}^-$ are defined as follows

$$
\begin{aligned}
&\tilde{f}^+(t, xva, \tilde{z}, \tilde{z}^I, \tilde{z}^C; \beta, \hat{V}, \hat{Z}) \\
&= -\Big(r_f^+ \big(xva - \frac{\mathbb{1}_{\{\tilde{z}+\hat{Z}>0, \beta=1\}}}{\sigma}(\tilde{z} + \hat{Z}) + \tilde{z}^I + \tilde{z}^C + (1-\alpha)\hat{V}\big)^+ \\
&\quad - r_f^- \big(xva - \frac{\mathbb{1}_{\{\tilde{z}+\hat{Z}>0, \beta=1\}}}{\sigma}(\tilde{z} + \hat{Z}) + \tilde{z}^I + \tilde{z}^C + (1-\alpha)\hat{V}\big)^- \\
&\quad + \frac{r_D - r_r^-(1 - \mathbb{1}_{\{\tilde{z}+\hat{Z}>0, \beta=1\}})}{\sigma}(\tilde{z} + \hat{Z})^+ + r_c^+(\alpha \hat{V}_t)^+ - r_c^-(\alpha \hat{V}_t)^- \\
&\quad - \frac{r_D - r_r^+(1 - \mathbb{1}_{\{\tilde{z}+\hat{Z}>0, \beta=1\}})}{\sigma}(\tilde{z} + \hat{Z})^- - r_D \tilde{z}^I - r_D \tilde{z}^C \Big) + r_D \hat{V}_t,
\end{aligned}
$$

$$
\tilde{f}^-(t, xva, \tilde{z}, \tilde{z}^I, \tilde{z}^C; \beta, \hat{V}, \hat{Z}) := -\tilde{f}^+(t, -xva, -\tilde{z}, -\tilde{z}^I, -\tilde{z}^C; \beta, -\hat{V}, -\hat{Z}).
$$

XVA BSDE

Theorem

In a filtered probability process $(\Omega, (\mathscr{F}_t)_{t\geq 0}, \mathscr{F}, \mathbb{P})$ *satisfying the usual conditions, where* $\mathscr{F}_t = \sigma(W_s, \beta_s, \varpi_s^I, \varpi_s^C : s \leq t)$*, the XVA BSDEs admit unique solutions* $(XVA^{\pm}, \tilde{Z}^{\pm}, \tilde{Z}^{I,\pm}, \tilde{Z}^{C,\pm})$*, given by*

$$XVA_t^{\pm} = XVA_\tau^{\pm} + \int_t^\tau \tilde{f}^{\pm}(s, XVA_s^{\pm}, \tilde{Z}_s^{\pm}, \tilde{Z}_s^{I,\pm}, \tilde{Z}_t^{C,\pm}; \beta, \hat{V}, \hat{Z})ds$$
$$- \int_t^\tau \tilde{Z}_s^{+} dW_s^{\mathbb{Q}} - \int_t^\tau \tilde{Z}_s^{I,\pm} d\varpi_t^{I,\mathbb{Q}} - \int_t^\tau \tilde{Z}_s^{C,\pm} d\varpi_t^{C,\mathbb{Q}}.$$

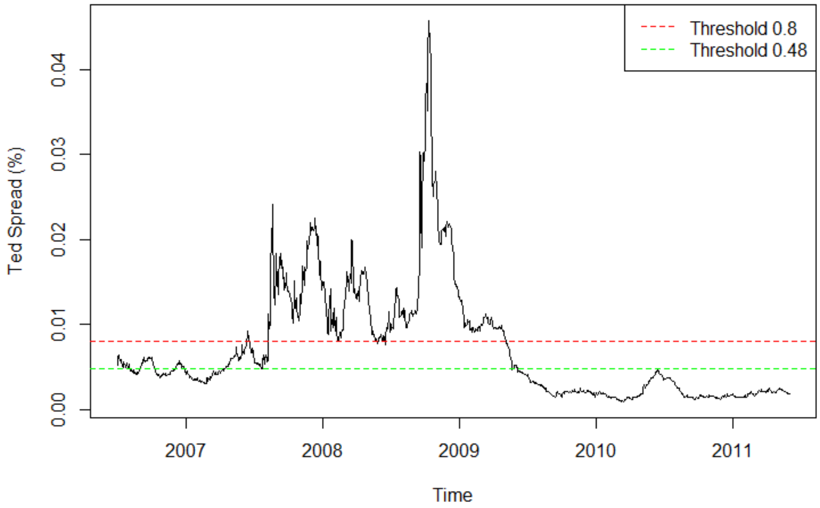
Simulation

Definition

The TED spread is the difference between three-month treasury bill rate and three-month London interbank offered rate (LIBOR) based on US dollars.

Boudt, Paulus, and Rosenthal claimed there is a two-regimes model of the TED spread with threshold 0.48 basis points.

Simulation

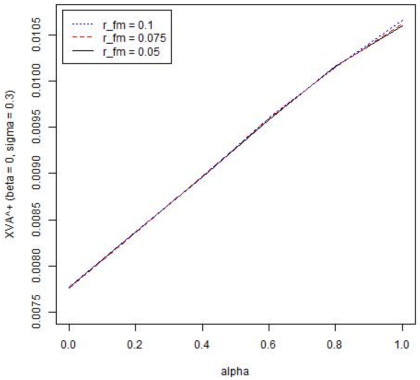

Threshold 0.8
Threshold 0.48
Ted Spread (%)
0.00
0.01
0.02
0.03
0.04
2007
2008
2009
2010
2011
Time

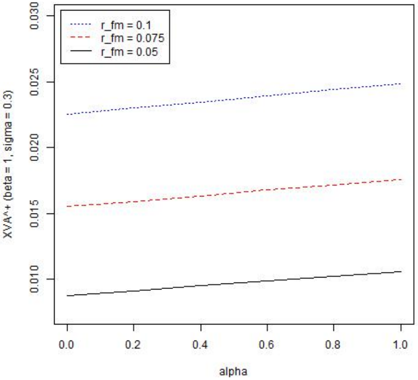
Simulation

	Normal Status	Crisis Status
Number of Statuses	2	2
Average Length (days)	507	361
Estimates of λ_U and λ_V	1.39	0.99

Table: Estimations of λ_U & λ_V.

Simulation





XVA^{+} for $\beta_t = 0$ (left) and $\beta_t = 1$ (right)

Simulation

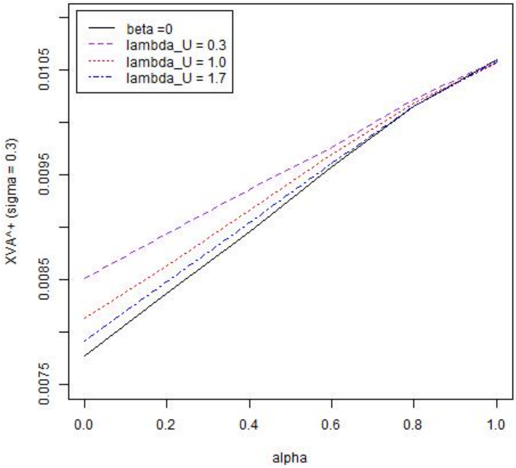


Figure: XVA^+ for $\beta_t = 0$ and a dynamic beta process.

Future Work

- Option pricing with European Repo market
- Feynman-Kac theorem of BSDEs with non-independent increment processes
- Comparison Theorem of BSDEs with non-independent increment processes
- Application of the alternating renewal processes in other pricing models
- extension of the alternating renewal process to a multi-statuses switching process

References

[1] Bichuch, M., Capponi, A., & Sturm, S. (2018). Arbitrage - free XVA. Mathematical Finance, 28(2), 582-620.

[2] Boudt, K., Paulus, E., & Rosenthal, D. (2013). Funding liquidity, market liquidity and TED spread: A two-regime model (No. 244). National Bank of Belgium.

[3] Cheridito, P., & Nam, K. (2017). BSE's, BSDE's and fixed-point problems. The Annals of Probability, 45(6A), 3795-3828.

[4] Crépey, S., & Song, S. (2015). BSDEs of counterparty risk. Stochastic Processes and their Applications, 125(8), 3023-3052.

[5] Gorton, G., & Metrick, A. (2012). Securitized banking and the run on repo. Journal of Financial economics, 104(3), 425 – 451.

[6] Hördahl, P., & King, M. R. (2008). Developments in repo markets during the financial turmoil. BIS Quarterly Review, 37.

[7] Métivier, M. (2011). Semimartingales: a course on stochastic processes (Vol. 2). Walter de Gruyter.

[8] Weinan, E., Han, J., & Jentzen, A. (2017). Deep learning-based numerical methods for high-dimensional parabolic partial differential equations and backward stochastic differential equations. Communications in Mathematics and

Thank you for your attention.

Any questions?