A robust Bayesian analysis of some premium principles based on distorted bands of prior distributions.

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2. **Robust Bayesian Analysis**
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Bayesian decision framework

Prior problem

- $X$ an observation from a distribution $P_{\theta}$ with density $p_{\theta}(x)$.
- $\theta$ is a parameter in the parameter space $\Theta$.
- $\pi(\theta)$ is a prior distribution over the set of states $\Theta$.
- $l(\theta|x)$ is the likelihood function of a particular sample.

Posterior problem

- $\pi_x(\theta)$ denotes the posterior density when $x$ is observed.

$$
\pi_x(\theta) = \frac{f(x|\theta)\pi(\theta)}{\int_\Theta f(x|\theta)\pi(\theta)\,d\theta} = \frac{f(x|\theta)\pi(\theta)}{m_\pi(x)}
$$

- $L$ is a loss function and $a \in A$ a set of alternatives.
- The posterior expected loss:

$$
\rho(a, L, \pi) = \frac{\int_\Theta L(a, \theta)p_{\theta}(x)\pi(\theta)d\theta}{m_\pi(x)} = E^{\pi_x}[L(a, \theta)].
$$
Problem: To make decision (inference)

Using the posterior distribution $\pi_x(\theta)$ in order to obtain some posterior quantity of interest.

For example:

- It is usual to use the posterior distribution to make predictions or inference.
- It is possible to minimize the posterior expected loss to make inference, obtaining the Bayes actions:

$$\text{Bayes Action } a^*_x(L, \pi) = \arg\min_{a \in A} \rho(a, L, \pi) = \arg\min_{a \in A} E^{\pi_x} [L(a, \theta)].$$
Robustness

The classical criticisms: A unique prior?

A Bayesian analysis is robust if it does not depend sensitively on the assumptions and calculation inputs on which it is based.

Imprecision in beliefs and preferences: The classical criticism

- Beliefs modelled by a class of prior distributions, $\Gamma$.

Example (Imprecise in beliefs: Given a prior belief $\pi$.)

Some classical prior classes:

$$\Gamma_\epsilon = \{\pi' : \pi' = (1 - \epsilon)\pi + \epsilon Q, Q \in Q\}. \text{ The } \epsilon\text{-contamination.}$$

$$\Gamma_{F_L,F_U,\pi} = \{\pi' : F_U(\theta) \geq F_{\pi'}(\theta) \geq F_L(\theta), \forall \theta \in \Theta\}. \text{ The band class.}$$
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A Bayesian analysis is robust if it does not depend sensitively on the assumptions and calculation inputs on which it is based.

A class of priors $\Gamma$ implies a class of posterior distributions $\Gamma_x$. Ordering Bayes action (by using a class of loss functions) or any quantity of interest. To study the range of some posterior quantity of interest. Sometimes it is difficult in practice. We will use a new class of priors.
"New Classes of Priors Based on Stochastic Orders and Distortion Functions"

J. Pablo Arias Nicolás
Fabrizio Ruggeri
Alfonso Suárez Llorens

It is needed

- Stochastic orders.
- Distortion functions.
Preliminaries: distortion functions


A distortion function $h$ is a non-decreasing continuous function $h : [0, 1] \rightarrow [0, 1]$ such that $h(0) = 0$ and $h(1) = 1$.

Distorted random variables: we represent a perturbation

Let $h$ be a distortion function and let $\pi(\theta)$ a specific prior belief. Let us consider the transformation:

$$F_{\pi_h}(\theta) = h \circ F_{\pi}(x) = h[F_{\pi}(x)].$$

- $F_{\pi_h}$ is also a distribution function for $\theta$ “a distorted random variable”.
- $F_{\pi_h}$ represents a perturbation of the underlying accumulated probability: to measure the uncertainty about $\theta$. 
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Some examples of distortion functions.

**Distortion functions**

- The classical power functions:
  \[ h_1(x) = 1 - (1 - x)^\alpha \text{ and } h_2(x) = x^\alpha, \quad \forall \alpha > 1. \]
  \[ F_{\pi h_1} (\theta) = 1 - (1 - F_\pi (\theta))^\alpha \text{ and } F_{\pi h_2} (\theta) = (F_\pi (\theta))^\alpha. \]

- The truncated distortions:
  \[ h_1(x) = \min\left\{ \frac{x}{\alpha}, 1 \right\} \text{ and } h_2(x) = \max\left\{ \frac{x - \alpha}{1 - \alpha}, 0 \right\}, \quad \forall 0 < \alpha < 1. \]
  \[ \pi_{h_1} = st \left[ \pi \mid \pi \leq F_\pi^{-1}(\alpha) \right] \text{ and } \pi_{h_2} = st \left[ \pi \mid \pi > F_\pi^{-1}(\alpha) \right]. \]

- The skewed distributions: \( \pi \) a symmetric distribution around 0.
  \[ h_{\pi \pi^\alpha} (x) = \int_{-\infty}^{F_\pi^{-1}(x)} 2\pi(\theta)F_\pi (\alpha\theta) d\theta, \text{ convex } \alpha > 0, \text{ concave } \alpha < 0. \]
  \[ \pi^\alpha (\theta) = 2\pi(\theta)F_\pi (\alpha\theta), \text{ right skewed } \alpha > 0, \text{ left skewed } \alpha < 0. \]
The likelihood ratio order: Shaked and Shanthikumar (2007)

Let $X$ and $Y$ be absolutely continuous [discrete] random variables with distribution functions $F_X$ and $F_Y$ and densities [discrete densities] $f_X$ and $f_Y$, respectively, such that

$$\frac{f_Y(t)}{f_X(t)}$$

increases over the union of the supports of $X$ and $Y$,

(here $a/0$ is taken to be equal to $\infty$ whenever $a > 0$). Then $X$ is said to be smaller than $Y$ in the likelihood ratio order (denoted by $X \leq_{lr} Y$).
Lemma: The relationship between \( \pi \) and \( \pi_h \).

Let \( \pi \) be a specific prior belief with distribution function \( F_\pi \) (absolutely continuous or discrete). Let \( h \) be a distortion function and \( F_{\pi_h}(\theta) = h \circ F_\pi(\theta) \). Then

- If \( h \) is convex then \( \pi \preceq_{lr} \pi_h \),
- If \( h \) is concave then \( \pi \succeq_{lr} \pi_h \).
A new class of priors

Definition

Let $\pi$ be a specific prior belief. We will define the distorted band $\Gamma_{h_1,h_2,\pi}$ associated with $\pi$ based on $h_1$ and $h_2$, a concave distortion function and a convex distortion function, respectively, (distorted band, for short), as

$$\Gamma_{h_1,h_2,\pi} = \{ \pi' : \pi_{h_1} \leq_{lr} \pi' \leq_{lr} \pi_{h_2} \} .$$

Remark

- A particular "neighborhood" band of $\pi \in \Gamma_{h_1,h_2,\pi}$.
- The lower and upper bound distributions are given by the distorted distributions.
- Uncertainty can be introduced just through an upper (lower) bound by considering $h_1$ ($h_2$) the identity function.
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- Uncertainty can be introduced just through an upper (lower) bound by considering $h_1$ ($h_2$) the identity function.
The posterior consequences

Main results

Finally, they conclude:

- The set of posterior distributions can be interpreted as a posterior distorted band:
  \[ \Gamma_{h_1,h_2,\pi,x} = \{ \pi' : \pi' \in \Gamma_{h_1,h_2,\pi} \} \longrightarrow \pi_{h_1,x} \leq lr \pi_x' \leq lr \pi_{h_2,x}. \]

- Let \( \pi \) be a specific prior belief and let \( \Gamma_{h_1,h_2,\pi} \) be the corresponding distorted band. Then
  \[ a^*(L,\pi_{h_1}) \leq a^*(L,\pi') \leq a^*(L,\pi_{h_2}), \]
  \[ \forall L \in \mathcal{L}_{sm} \text{ and } \forall \pi' \in \Gamma_{h_1,h_2,\pi}. \]

\( \mathcal{L}_{sm} \) is considered as the class of all convex loss functions which satisfy the submodularity. Widely used loss functions in the literature are included in this class: quadratic, lineal, mean square, . . .
The insurance risk $X_\theta \sim F(x, \theta)$ is a random variable which depends on a parameter $\theta \in \Theta$.

Example

Let be a insurance risk $X_\theta$ which follows a Poisson distribution, $X \sim P(\theta)$. 
Insurance risk

The Premiums

Given a risk \( X_\theta \), a premium principle is a functional \( H[X] \) that maps \( X \) to a non-negative real number, which is the premium charged to the policyholder to compensate the insurer for bearing the risk \( X \).

Example

- Net Premium: \( H[X_\theta] = E[X_\theta] \).
- Dutch Premium: \( H[X_\theta] = E[X_\theta] + \beta_1 E[(X_\theta - \beta_2 E[X_\theta])^+] \), \( \beta_1 > 0, \beta_2 > 0 \).
- Exponential utility premium: \( H[X_\theta] = \frac{1}{\beta} \log E[e^{\beta X_\theta}] \), \( \beta > 0 \).
- Esscher premium: \( H[X_\theta] = \frac{E[X_\theta e^{\beta X_\theta}]}{E[e^{\beta X_\theta}]} \), \( \beta > 0 \).
The Bayesian point of view:

1. The parameter follows a prior belief, $\pi$.
2. Based on a sample $x$, we obtain the posterior density $\pi_x$.

The Risk Premium

As it is clear that $H[X_\theta]$ inherits the dependency of the parameter, $H[X_\theta] = P_{R,H}(\theta)$ is known as the risk premium based on $H$.

Example

Let be an insurance risk $X_\theta$ which follows a Poisson distribution, $X \sim P(\theta)$. Given the net premium $H[X_\theta] = E[X_\theta]$, then the risk premium is $P_{R,H}(\theta) = \theta$. 
Defining Bayesian premiums in risk theory

The Bayesian point of view:

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The Risk Premium

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Example

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How to compute premiums from the Bayesian perspective?

- If we consider the random risk $P_{R,H}(\pi)$ as a transformation of the prior belief distribution $\Rightarrow$ We compute the collective premium $H'[P_{R,H}(\pi)] = P_{C,H,H'}(\pi)$.
- If we consider the random risk $P_{R,H}(\pi_x)$ as a transformation of the posterior belief distribution $\Rightarrow$ We compute the Bayesian premium $H'[P_{R,H}(\pi_x)] = P_{B,H,H'}(\pi_x)$.

Example

Let $X_\theta$ be an insurance risk which follows a Poisson distribution, $X \sim P(\theta)$. Given the net premium $H[X_\theta] = E[X_\theta]$, then $P_{R,H}(\theta) = \theta$. Let suppose that $\pi \sim Ga(a,b)$ and $\pi_x \sim Ga(a + n\bar{x}, b + n)$, from a sample. Then, given the net premium for $H'[X_\theta]$, we obtain

$$P_{C,H,H'}(\pi) = \frac{a}{b} \quad \text{and} \quad P_{B,H,H'}(\pi_x) = \frac{a + n\bar{x}}{b + n}.$$
Some new definitions

Positive dependence between $\pi$ and $X$ can be expected

Given a premium principle $H$, we will say that $X_\theta$ is increasing in risk for $H$, in short $\text{IR}_H$, if the Risk premium $P_{R,H}(\theta)$ is non-decreasing in the parameter space $\Theta$.

A class of premium principle

Given $X$ and $Y$ two random risks we will denote by $\mathcal{H}_{lr}$ the class of all premium principles such that preserve the likelihood ratio order:

$$\mathcal{H}_{lr} = \{ H : \text{ If } X \leq_{lr} Y \text{ then } H[X] \leq H[Y] \}.$$
Main results

Lemma

Given $\theta_1 < \theta_2$, if $X_{\theta_1} \leq_{lr} X_{\theta_2}$ then $X_\theta$ is $\text{IR}_H$ for all $H \in \mathcal{H}_{lr}$.

Some members of the class

- The net premium.
- The Esscher premium.
- The Dutch premium
- The exponential utility premium.
- The Wang premium.
- The generalized variance premium.
Main results

Lemma

Given $\theta_1 < \theta_2$, if $X_{\theta_1} \leq_{lr} X_{\theta_2}$ then $X_\theta$ is $I_{RH}$ for all $H \in \mathcal{H}_{lr}$.

Some members of the class

- The net premium.
- The Esscher premium.
- The Dutch premium
- The exponential utility premium.
- The Wang premium.
- The generalized variance premium.
Main results

Theorem

Let $X_\theta$ and $\pi$ be a random risk depending on a parameter $\theta$ and a specific prior belief, respectively. Let $\Gamma_{h_1,h_2,\pi}$ be the corresponding distorted band associated with $\pi$ based on $h_1$ and $h_2$. Then

1. $P_{C,H,H'}(\pi_{h_1}) \leq P_{C,H,H'}(\pi') \leq P_{C,H,H'}(\pi_{h_2}),$

2. $P_{B,H,H'}(\pi_{h_1,x}) \leq P_{B,H,H'}(\pi'_{x}) \leq P_{B,H,H'}(\pi_{h_2,x}),$

for all premium principle $H$ such that $X_\theta$ is $IR_H$, for all $H' \in \mathcal{H}_{lr}$ and for all $\pi' \in \Gamma_{h_1,h_2,\pi}$. 
Example 1: The problem

"The Esscher premium principle in risk theory: a Bayesian sensitivity study"

E. Gómez Deniz
A. Hernández Bastida
F. J. Vázquez Polo

- To use the $\epsilon$-contamination:

$$\Gamma_\epsilon = \{ \pi' : \pi' = (1 - \epsilon)\pi + \epsilon Q, Q \in Q_1 \},$$

where $Q_1 = \{ \text{All probability distribution} \}$ and $Q_2 = \{ \text{All distributions which are unimodal with the same mode, } \theta_0, \text{ as that of } \pi_0 \}$.  

- To use the Esscher premium as the collective and the Bayesian premiums.
Example 1: The model

The initial problem

- The total number of claims is a Poisson distribution $P(\theta)$, $IR_H \forall H \in \mathcal{H}_{lr}$.
- The amount of an individual claim is taken as 100 monetary units.
- The prior distribution follows a Gamma distribution $\Gamma(5, 2)$ -the actuary expects 5 claims every 2 years, 2.5 claims-.
- We know the sample mean, $\bar{x}$, of the number of claims for 10 periods.
- The posterior distribution follows a Gamma distribution $\Gamma(5 + 10\bar{x}, 2 + 10)$.
- For the collective and Bayesian premium we will consider the following combinations of $H$ and $H^*$:

<table>
<thead>
<tr>
<th>$H - H^*$</th>
<th>Net-Net</th>
<th>Esscher-Net</th>
<th>Esscher-Esscher</th>
<th>Exponential Utility-Net</th>
</tr>
</thead>
<tbody>
<tr>
<td>Collective Premium</td>
<td>$\frac{a}{b}$</td>
<td>$e^{\beta} \frac{a}{b}$</td>
<td>$\frac{e^{\beta}a}{b - \beta e^{\beta}}$</td>
<td>$(e^{\beta} - 1) \frac{a}{b^2}$</td>
</tr>
<tr>
<td>Bayesian Premium</td>
<td>$\frac{a + nx}{b + n}$</td>
<td>$e^{\beta} \frac{a + nx}{b + n}$</td>
<td>$\frac{e^{\beta}a + nx}{(b + n) - \beta e^{\beta}}$</td>
<td>$(e^{\beta} - 1) \frac{a + nx}{b(b + n)}$</td>
</tr>
</tbody>
</table>

Table: Values for the premiums depending on the premium principles.

$(\beta = 0.0953$ is the risk aversion constant).
The model

The distorted problem

- There are not closed-form expressions for the bounds, neither for the prior bounds $\pi_{h_{\alpha_1}}(\theta)$ and $\pi_{h_{\alpha_2}}(\theta)$ nor for the posterior ones $\pi_{h_{\alpha_1},x}(\theta)$ and $\pi_{h_{\alpha_2},x}(\theta))$. However, it is known that

$$P_{B,H,H^*}(\pi_{h_{\alpha_1},x}) \leq P_{B,H,H^*}(\pi'_{x}) \leq P_{B,H,H^*}(\pi_{h_{\alpha_1},x}), \forall \pi' \in \Gamma_{h_{\alpha_1},h_{\alpha_2},\pi},$$

- We simulate different scenarios: $\alpha = 1.05, 1.11, 1.15, 1.2$ and $\bar{x} = 2, 5$. 

Example: \( \bar{x} = 2 \) and \( n = 10 \)

**Net Premium - Net Premium**

- \( \alpha_1 = \alpha_2 = 1.05 \)
- \( \alpha_1 = \alpha_2 = 1.1 \)
- \( \alpha_1 = \alpha_2 = 1.15 \)
- \( \alpha_1 = \alpha_2 = 1.2 \)

**Esscher Premium - Esscher Premium**

- \( \alpha_1 = \alpha_2 = 1.05 \)
- \( \alpha_1 = \alpha_2 = 1.1 \)
- \( \alpha_1 = \alpha_2 = 1.15 \)
- \( \alpha_1 = \alpha_2 = 1.2 \)

**Esscher Premium - Net Premium**

**Exp. utility Premium - Net Premium**
Example: $\bar{x} = 5$ and $n = 10$

Net Premium - Net Premium

Esscher Premium - Esscher Premium

Esscher Premium - Net Premium

Exp. utility Premium - Net Premium

Thank you for your attention!

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