Optimal Reinsurance with Multiple Reinsurers

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(joint work with Tim J. Boonen)

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The Main Idea

- Let $\mathcal{X}$ be a given collection of random variables on a given measurable space (S, Σ), and $X \in \mathcal{X}$ the risk exposure of an insurer.

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- In a static optimal reinsurance model, the insurer seeks an indemnification $I(X)$ from a reinsurer, among a collection of feasible indemnities, in return for a reinsurance premium $\pi(I(X))$, so as to minimize

$$\rho\Big(X - I(X) + \pi(I(X))\Big),$$

where:

  - $\pi : \mathcal{X} \to \mathbb{R}$ is the reinsurance premium principle;
  - $\rho : \mathcal{X} \to \mathbb{R}$ is the insurer's risk measure.

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- Here, we extend this literature to the case of multiple reinsurers and heterogeneous beliefs, and we augment the recent work of Boonen and Ghossoub (2019).

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- The optimal reinsurance indemnities enable us to identify a representative reinsurer in both situations.

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- Let S be a state space, and let $X : S \to \mathbb{R}^+$ represent an exogenously given risk that we interpret as the insurer's loss at a given future reference time.
- Denote by Σ the σ-algebra $\sigma\{X\}$ generated by X on S.

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$\Longrightarrow$ We do not impose any restriction on how the beliefs $\mathbb{P}, \mathbb{Q}_1, \mathbb{Q}_2, \ldots, \mathbb{Q}_n$ can differ, hence allowing for much flexibility.

$\Longrightarrow$ In particular, any couple of beliefs is allowed to exhibit disagreement about zero-probability events, and, as an extreme case, to be mutually singular.

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- For $i \in \{1, 2, \ldots, n\}$, the distortion premium principle used by reinsurer i is given by

$$\pi^{\theta_i, T_i, \mathbb{Q}_i}(Y) := (1 + \theta_i) \int_0^\infty T_i\left(\mathbb{Q}_i(Y > z)\right) dz, \text{ for all } Y \in L^\infty_+(S, \Sigma, \mu),$$

where $\theta_i \geqslant 0$ is interpreted as a risk-loading charged by reinsurer i, and $T_i : [0, 1] \to [0, 1]$ is the distortion function used by reinsurer i.

Optimal Reinsurance with Multiple Reinsurers

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- The insurer seeks a reinsurance indemnification $f_i(X)$ from each reinsurer $i \in \{1, 2, \ldots, n\}$, in return for a premium payment of $\pi^{\theta_i, T_i, \mathbb{Q}_i}(f_i(X))$.

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- The insurer's total exposure after reinsurance is then given by

$$X - \sum_{i=1}^{n} f_i(X) + \sum_{i=1}^{n} \pi^{\theta_i, T_i, \mathbb{Q}_i}(f_i(X)).$$

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- The feasible indemnity functions are such that $\{f_i\}_{i=1}^{n} \subset \mathcal{F}$ and $\sum_{i=1}^{n} f_i \in \mathcal{F}$, where:

$$\mathcal{F} := \left\{ f : [0, M] \to [0, M] \,\middle|\, 0 \leqslant f(X) \leqslant X, \ \mu\text{-a.s.}; \ 0 \leqslant f'(z) \leqslant 1, \ \text{for a.e. } z \in [0, M] \right\}.$$

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- The insurer uses a distortion risk measure $\rho^{\mathbb{P}} : L^{\infty}_{+}(S, \Sigma, \mu) \to \mathbb{R}$, given by:

$$\rho^{\mathbb{P}}(Y) := \int_0^{\infty} T_0\left(\mathbb{P}(Y > z)\right) dz, \text{ for all } Y \in L^{\infty}_{+}(S, \Sigma, \mu),$$

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- Hence, the insurer's problem can be formulated as

Optimal Reinsurance with Multiple Reinsurers

$$\begin{array}{ll} \inf\limits_{\{f_i\}_{i=1}^n} & \rho^{\mathbb{P}}\left(X - \sum_{i=1}^n f_i(X) + \sum_{i=1}^n \pi^{\theta_i, T_i, \mathbb{Q}_i}(f_i(X))\right) \\ \text{s.t.} & \{f_i\}_{i=1}^n \subset \mathcal{F} \text{ and } \sum_{i=1}^n f_i \in \mathcal{F}. \end{array} \quad (1)$$

Optimal Reinsurance with Multiple Reinsurers

- Define the capacity $\upsilon : \Sigma \to \mathbb{R}^+$ by

$$\upsilon(B) := \min_{1 \leqslant i \leqslant n} \Big\{ (1+\theta_i)\, T_i\left(\mathbb{Q}_i(B)\right) \Big\}, \quad \forall\ B \in \Sigma.$$

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- Define, for each $z \in [0, M]$, the collection $\mathcal{I}(z) \subset \{1, 2, \ldots, n\}$ by

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$$\mathcal{I}(z) := \underset{1 \leqslant j \leqslant n}{\operatorname{argmin}} \left(1+\theta_j\right) T_j \Big(\mathbb{Q}_j(X > z)\Big).$$

- Then, for all $z \in [0, M]$,

$$i \in \mathcal{I}(z) \implies \left(1+\theta_i\right) T_i\left(\mathbb{Q}_i(X > z)\right) = \upsilon\left(X > z\right).$$

Optimal Reinsurance with Multiple Reinsurers

Theorem (Optimal Indemnity Profiles)

Profile $\{f_i\}_{i=1}^n$ *is a solution to Problem* (1) *if and only if for each* $x \in [0, M]$,
$f_i(x) = \int_0^x h_i(z)dz$, *where for* $i = 1, \ldots, n$, *and for a.e.* $z \in [0, M]$,

$$h_i(z) = \begin{cases} \gamma_i(z) & \text{if } \ i \in \mathcal{I}(z), \\ 0 & \text{otherwise}, \end{cases}$$

and for a.e. $z \in [0, M]$, $\gamma_i(z) \in [0, 1]$ *is such that*

$$\sum_{i=1}^n h_i(z) = \sum_{i \in \mathcal{I}(z)} \gamma_i(z) = \begin{cases} 1 & \text{if } \ \upsilon(X > z) < T_0\left(\mathbb{P}(X > z)\right), \\ \phi(z) & \text{if } \ \upsilon(X > z) = T_0\left(\mathbb{P}(X > z)\right), \\ 0 & \text{if } \ \upsilon(X > z) > T_0\left(\mathbb{P}(X > z)\right), \end{cases}$$

for some $\phi(z) \in [0, 1]$.

Optimal Reinsurance with Multiple Reinsurers

$\Longrightarrow$ This result provides a full characterization of all reinsurance indemnities that solve Problem (1).

$\Longrightarrow$ For all $z \in [0, M]$, marginal indemnities $h_i(z)$ are strictly positive only for the reinsurers that have the smallest value among $(1+\theta_1)\, T_1\left(\mathbb{Q}_1(X > z)\right), \ldots,$ and $(1+\theta_n)\, T_n\left(\mathbb{Q}_n(X > z)\right)$.

$\Longrightarrow$ This generalizes the results of Boonen et al. (2016), where it was assumed that $\mathbb{Q}_1 = \cdots = \mathbb{Q}_n = \mathbb{P}$, as well as Assa (2015) and Cui et al. (2013) who consider only one reinsurer.

Optimal Reinsurance with Multiple Reinsurers

$\Longrightarrow$ If profile $\{f_i\}_{i=1}^n$ is a solution to Problem (1), then there exists some $\phi : [0, M] \to [0, 1]$, such that the total indemnification received by the insurer for a loss of value $x \in [0, M]$ is

$$f(x) := \sum_{i=1}^{n} f_i(x) = \sum_{i=1}^{n} \int_0^x h_i(z)dz = \int_0^x \sum_{i=1}^{n} h_i(z)dz = \int_0^x \sum_{i \in \mathcal{I}(z)} \gamma_i(z)dz$$

$$= Leb\left(\mathcal{A}_x\right) + \int_{\mathcal{B}_x} \phi(z)dz,$$

where:

$$\mathcal{A}_x := \left\{ z \in [0, x] : \upsilon(X > z) < T_0\left(\mathbb{P}(X > z)\right) \right\};$$

$$\mathcal{B}_x := \left\{ z \in [0, x] : \upsilon(X > z) = T_0\left(\mathbb{P}(X > z)\right) \right\}.$$

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- This means that:
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  - The optimal coverage is then allocated among the n reinsurers in a way that minimizes the total cost of coverage.

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    - All reinsurers can be treated collectively by means of a hypothetical premium principle in order to determine the optimal total risk that is ceded to all reinsurers.
    - The optimal coverage is then allocated among the n reinsurers in a way that minimizes the total cost of coverage.
- This augments the findings of Boonen and Ghossoub (2019), who focus on expected-value premium principles for the reinsurers, but a general (monotone) objective function for the insurer.

Representative Reinsurer

Definition

Define the functional $\pi : L^{\infty}_{+}(S, \Sigma, \mu) \to \mathbb{R}$ by:

$$\pi(Y) := \int_0^{\infty} \upsilon(Y > z)dz, \text{ for all } Y \in L^{\infty}_{+}(S, \Sigma, \mu), \tag{2}$$

where υ is as previously defined:

$$\upsilon(Y > z) := \min_{1 \leqslant i \leqslant n} \left\{ (1 + \theta_i) T_i (\mathbb{Q}_i(Y > z)) \right\}, \ \forall z \in [0, M].$$

$\Longrightarrow$ This will be the "representative premium functional" of the n reinsurers.

Representative Reinsurer

Proposition (Cost-Efficiency of Optimal Profiles)

- *If profile* $\{f_i\}_{i=1}^n$ *is a solution to Problem* (1)*, then*

$$\sum_{i=1}^{n} \pi^{\theta_i, T_i, \mathbb{Q}_i}(f_i(X)) = \int_0^{f(M)} \upsilon(f(X) > z)dz = \pi\left(f\left(X\right)\right),$$

where $f := \sum_{i=1}^n f_i$.

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Proposition (Cost-Efficiency of Optimal Profiles)

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where $f := \sum_{i=1}^n f_i$.

- *Moreover, for any feasible profile* $\{g_i\}_{i=1}^n$ *for Problem* (1) *that satisfies* $\sum_{i=1}^n g_i\left(X\right) = \sum_{i=1}^n f_i\left(X\right) = f\left(X\right)$, μ*-a.s., we have*

$$\sum_{i=1}^{n} \pi^{\theta_i, T_i, \mathbb{Q}_i}(g_i(X)) \geqslant \sum_{i=1}^{n} \pi^{\theta_i, T_i, \mathbb{Q}_i}(f_i(X)) = \pi\left(f\left(X\right)\right).$$

Representative Reinsurer

- Now, consider the following two problems:

$$\inf_{f \in \mathcal{F}} \quad \rho^{\mathbb{P}}\Big(X - f(X) + \pi(f(X))\Big), \tag{3}$$

where π is as in eq. (2) ("representative premium functional"); and

$$\begin{aligned} \inf_{\{f_i\}_{i=1}^n} \quad & \textstyle\sum_{i=1}^n \pi^{\theta_i, T_i, \mathbb{Q}_i}(f_i(X)) \\ \text{s.t.} \quad & \{f_i\}_{i=1}^n \subset \mathcal{F}, \text{ and } \textstyle\sum_{i=1}^n f_i(X) = f(X), \ \mu\text{-a.s.} \end{aligned} \tag{4}$$

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- For each $f \in \mathcal{F}$, define the collection

$$\mathcal{F}(f) := \Big\{\{f_i\}_{i=1}^n \subset \mathcal{F} : \{f_i\}_{i=1}^n \text{ solves Problem (4) for the given } f\Big\}.$$

Representative Reinsurer

Theorem (Cost-Minimizing Profiles)

Fix $f \in \mathcal{F}$. Then, $\{f_i\}_{i=1}^n \in \mathcal{F}(f)$ if and only if the following two conditions hold simultaneously:

(i) $\{f_i\}_{i=1}^n$ *is such that for each* $i \in \{1, 2, \ldots, n\}$ *and for each* $x \in [0, M]$,
$f_i(x) = \int_0^x h_i(z)dz$, *where for a.e.* $z \in [0, M]$,

$$h_i(z) = 0 \quad \textit{whenever} \quad i \notin \mathcal{I}(z);$$

(ii) $\{f_i\}_{i=1}^n \subset \mathcal{F}$ *and* $\sum_{i=1}^n h_i(z) = f'(z)$, *for a.e.* $z \in [0, M]$.

Hence, reinsurance contracts in $\mathcal{F}(f)$ are determined by means of a characterization of $\{f_i'(z)\}_{i=1}^n$ for a.e. $z \in [0, M]$.

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Theorem (Representative Reinsurer)

The following are equivalent:

(i) $\{f_i\}_{i=1}^n$ *is optimal for Problem* (1);

(ii) $\sum_{i=1}^n f_i$ *is optimal for Problem* (3) *with* π *as in eq.* (2), *and* $\{f_i\}_{i=1}^n \in \mathcal{F}\left(\sum_{j=1}^n f_j\right)$.

Budget-Constrained Optimal Reinsurance with Multiple Reinsurers

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- The insurer's problem becomes:

$$
\begin{aligned}
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\text{s.t.} \quad & \textstyle\sum_{i=1}^{n} \pi^{\theta_i, T_i, \mathbb{Q}_i}\big(f_i(X)\big) \leqslant p, \\
& \{f_i\}_{i=1}^{n} \subset \mathcal{F}, \;\; \textstyle\sum_{i=1}^{n} f_i \in \mathcal{F},
\end{aligned}
\tag{5}
$$

where $p \geqslant 0$ is an exogenously given premium budget.

Budget-Constrained Optimal Reinsurance with Multiple Reinsurers

- If there exists a profile $\{f_i^*\}_{i=1}^n$ solving Problem (1) such that $\sum_{i=1}^n \pi^{\theta_i, T_i, \mathbb{Q}_i}(f_i^*(X)) \leqslant p$, then the profile $\{f_i^*\}_{i=1}^n$ also solves Problem (5).

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- Hence, we define

$$\bar{\pi} := \inf \left\{ \sum_{i=1}^n \pi^{\theta_i, T_i, \mathbb{Q}_i}(f_i^*(X)) : \{f_i^*\}_{i=1}^n \textit{ solves Problem } (1) \right\}.$$

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- If $p \geqslant \bar{\pi}$, then there exists a solution to the unconstrained Problem (1) that is also feasible in the constrained Problem (5). Thus, this will also be an optimal solution for Problem (5).

- Therefore, we proceed in this section with the more challenging problem of finding solutions to Problem (5) when $p \in [0, \bar{\pi})$.

Budget-Constrained Optimal Reinsurance with Multiple Reinsurers

- Since the objective and the premium budget constraint in Problem (5) are linear in $\{f_i\}_{i=1}^{n}$, using standard techniques, we introduce a Lagrange multiplier $\lambda \geqslant 0$ associated with the constraint in Problem (5).

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- Hence, we consider the following auxiliary problem:

$$\begin{aligned} \inf_{\{f_i\}_{i=1}^n} \quad & \rho^{\mathbb{P}}\left(X - \sum_{i=1}^n f_i(X) + \sum_{i=1}^n \pi^{\theta_i, T_i, \mathbb{Q}_i}(f_i(X))\right) + \lambda \left(\sum_{i=1}^n \pi^{\theta_i, T_i, \mathbb{Q}_i}(f_i(X)) - p\right) \\ \text{s.t.} \quad & \{f_i\}_{i=1}^n \subset \mathcal{F} \ \text{ and } \ \sum_{i=1}^n f_i \in \mathcal{F}. \end{aligned} \tag{6}$$

Budget-Constrained Optimal Reinsurance with Multiple Reinsurers

Lemma

If there exists $\lambda^* \geqslant 0$ *such that:*

(i) $\{f^*_{i,\lambda*}\}_{i=1}^n$ *is optimal for Problem* (6) *with* $\lambda = \lambda^*$*; and,*

(ii) $\sum_{i=1}^n \pi^{\theta_i, T_i, \mathbb{Q}_i}(f_{i,\lambda*}(X)) = p$,

then $\{f^*_{i,\lambda}\}_{i=1}^n$ *is optimal for Problem* (5).

Budget-Constrained Optimal Reinsurance with Multiple Reinsurers

Theorem (Optimal Indemnity Profiles)

For a given $\lambda \geqslant 0$*, profile* $\{f^*_{i,\lambda}\}_{i=1}^n$ *solves Problem* (6) *with* $p \in [0, \bar{\pi})$ *if and only if for each* $x \in [0, M]$*,* $f^*_{i,\lambda}(x) = \int_0^x h^*_{i,\lambda}(z)dz$*, where for* $i = 1, \ldots, n$*, and for a.e.* $z \in [0, M]$*,*

$$h^*_{i,\lambda}(z) = \begin{cases} \gamma_{i,\lambda}(z) & \text{if } \; i \in \mathcal{I}(z), \\ 0 & \text{otherwise}, \end{cases}$$

and for a.e. $z \in [0, M]$*,* $\gamma_{i,\lambda}(z) \in [0, 1]$ *is such that*

$$\sum_{i=1}^n h^*_{i,\lambda}(z) = \sum_{i \in \mathcal{I}(z)} \gamma_{i,\lambda}(z) = \begin{cases} 1 & \text{if } \; (1+\lambda)\upsilon(X > z) < T_0\left(\mathbb{P}(X > z)\right), \\ \phi_\lambda(z) & \text{if } \; (1+\lambda)\upsilon(X > z) = T_0\left(\mathbb{P}(X > z)\right), \\ 0 & \text{if } \; (1+\lambda)\upsilon(X > z) > T_0\left(\mathbb{P}(X > z)\right), \end{cases}$$

for some $\phi_\lambda(z) \in [0, 1]$*.*

Moreover, there exists $\lambda^* > 0$ *such that* $\sum_{i=1}^n \pi^{\theta_i, T_i, \mathbb{Q}_i}(f^*_{i,\lambda^*}(X)) = p$*.*

Representative Reinsurer with a Budget Constraint

- The existence of a representative reinsurer can be shown similarly to the unconstrained case (see paper).
- The same "representative premium functional" is used.

Conclusion

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Conclusion

- We present a unifying theory of optimal reinsurance design, where there are finitely many reinsurers, distortion risk measures, and heterogeneous beliefs regarding the underlying distribution.
- Optimal indemnity contracts have a layer-type shape.
- Moreover, we characterize the optimal contracts in the presence of a budget constraint, i.e., an upper bound constraint on the aggregate premium.
- For both the unconstrained and the budget-constrained reinsurance problems, we show the existence of a representative reinsurer.

Thank You