Bilateral Risk Sharing with no Aggregate Uncertainty under RDU

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UNIVERSITY OF AMSTERDAM

To Bet or Not to Bet... In EUT

- When is it Pareto-optimal for risk-averse agents to take bets?

  $\Longrightarrow$ Starting from an economic environment with no aggregate uncertainty, under what conditions is it Pareto-improving to introduce uncertainty in the economy through betting (trade of an uncertain asset)?

- One obvious case is when the agents are risk-averse EU-maximizers and do not share beliefs (Billot et al., 2000, ECMA):

  $\Longrightarrow$ If the agents disagree on probability assessments, then they find it Pareto-improving to engage in uncertainty-generating trade (i.e., to bet):

$$\text{Disagreement about beliefs} \overset{EUT}{\Longrightarrow} \text{Betting is Pareto-improving}$$

  $\Longrightarrow$ Conversely, disagreement about probabilities is the only way that betting may be Pareto-improving when starting from a no-betting allocation:

$$\text{Common beliefs} \overset{EUT}{\Longrightarrow} \text{Betting is not Pareto-improving (no-betting PO)}$$

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Bilateral Risk Sharing: The Main Idea

- We examine a situation in which both the agent and the counterparty are RDU, with different distortions of the same underlying probability measure.

$\Longrightarrow$ We show that, as long as the agents' distortion functions satisfy a certain consistency requirement, PO allocations are no-betting allocations.

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Setting

- Let (S, Σ) be a measurable space, and let $B(\Sigma)$ be the vector space of all bounded, $\mathbb{R}$-valued, and Σ-measurable functions on (S, Σ).

- There are two agents who seek a risk-sharing arrangement.

- Agent 1 is subject to a given risk $X_1 \in B(\Sigma)$ and Agent 2 is subject to a risk $X_2 \in B(\Sigma)$, where the realizations are interpreted as losses.

- We assume no aggregate uncertainty in this economy, which implies that $X_1 + X_2 = c$, for an exogenously given $c \in \mathbb{R}$.

$\Longrightarrow$ **Trading is therefore seen as betting rather than as hedging**.

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- An **allocation** is a pair $\left(\hat{X}_1, \hat{X}_2\right) \in B(\Sigma) \times B(\Sigma)$ such that $\hat{X}_1 + \hat{X}_2 = X_1 + X_2 = c$.
- An allocation $\left(\hat{X}_1, \hat{X}_2\right) \in B(\Sigma) \times B(\Sigma)$ is called a **no-betting allocation** if $\hat{X}_i(s) = \hat{X}_i(s')$, for all $s, s' \in S$, and for $i = 1, 2$.

  $\Longrightarrow$ For example, $(\alpha c, (1-\alpha) c)$ is a no-betting allocation, for any $\alpha \in \mathbb{R}$.
- Agent 1 has initial wealth $W_0^1 \in \mathbb{R}$, and his/her total state-contingent wealth after risk sharing is the random variable $W^1 \in B(\Sigma)$ defined by
$$W^1(s) := W_0^1 - \hat{X}_1(s), \ \forall s \in S.$$
- Agent 2 has initial wealth $W_0^2 \in \mathbb{R}$, and his/her total state-contingent wealth after risk sharing is the random variable $W^2 \in B(\Sigma)$ defined by
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Assumptions

- The preferences of Agent 1 are to maximize:

$$\int \hat{u}_1\left(W^1\right)\, dT_1 \circ P = \int \hat{u}_1\left(W_0^1 - \hat{X}_1\right)\, dT_1 \circ P.$$

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- The utility functions $\hat{u}_i$ are increasing, strictly concave, continuously differentiable, and satisfy the Inada conditions $\lim_{x \to -\infty} \hat{u}_i'(x) = +\infty$ and $\lim_{x \to +\infty} \hat{u}_i'(x) = 0$.

- The probability weighting functions $T_i : [0,1] \to [0,1]$ are such that $T_i(0) = 0$, $T_i(1) = 1$, and functions T_i are absolutely continuous and increasing.

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What if there is aggregate uncertainty?

- In part, an open question...
- Chateauneuf et al. (2000), Tsanakas and Christofides (2006), Carlier and Dana (2008), Chakravarty and Kelsey (2015) all assume that the probability weighting functions are convex.
- Xia and Zhou (2016) assume that all agents use the same probability weighting function.
- Jin et al. (2019) show that Pareto optimal risk-sharing contracts exist under technical conditions that require aggregate market uncertainty.
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Reminiscent problem

- Define $Y := \hat{X}_1 - c$;
- Define the utility functions u_1 and u_2 by

$$u_1(x) := \hat{u}_1(W_0^1 - c + x) \text{ and } u_2(x) := \hat{u}_2(W_0^2 + x), \text{ for } x \in \mathbb{R}.$$

- A risk-sharing contract $Y^* \in B(\Sigma)$ is **Pareto optimal (PO)** if there does not exist any other risk-sharing contract $Y \in B(\Sigma)$ such that

$$\int u_1(-Y)\, dT_1 \circ P \geqslant \int u_1(-Y^*)\, dT_1 \circ P \text{ and } \int u_2(Y)\, dT_2 \circ P \geqslant \int u_2(Y^*)\, dT_2 \circ P,$$

with at least one strict inequality.

- If $Y^* \in B(\Sigma)$ is PO, we say that the allocation $\left(\hat{X}_1^*, \hat{X}_2^*\right)$ is PO, where $\hat{X}_1^* := Y^* + c$ and $\hat{X}_2^* := -Y^*$.

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$$\left(\widehat{\mathcal{P}}_{V_0}\right) \qquad \sup_{Y \in B(\Sigma)} \left\{ \int u_1\left(-Y\right)\, dT_1 \circ P : \int u_2\left(Y\right)\, dT_2 \circ P \geqslant V_0 \right\}.$$

Lemma

(i) *If the risk-sharing contract $Y^* \in B\left(\Sigma\right)$ is Pareto optimal, then it solves Problem $\left(\widehat{\mathcal{P}}_{V_0}\right)$ with $V_0 := \int u_2\left(Y^*\right)\, dT_2 \circ P$;*

(ii) *for a given $V_0 \in \mathbb{R}$, any solution to Problem $\left(\widehat{\mathcal{P}}_{V_0}\right)$ is Pareto optimal;*

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Optimal Risk-Sharing Between Two RDU Agents

Theorem

A risk-sharing contract Y^* is PO if there exists some $\lambda^* > 0$ such that

$$Y^* = m^{-1}\left(\lambda^* \delta'\left(T_1(U)\right)\right),$$

where:

- *U is a random variable on (S, Σ, P) with a uniform distribution on $(0,1)$;*
- *$m(x) := \frac{u_1'(-x)}{u_2'(x)}$, for all $x \geqslant 0$;*
- *δ is the convex envelope on $[0,1]$ of the function $\Psi : [0,1] \to \mathbb{R}$ defined by $\Psi(t) := \widetilde{T}_2\left(T_1^{-1}(t)\right)$, where $\widetilde{T}_2(t) = 1 - T_2(1-t)$, for each $t \in [0,1]$.*

Moreover, for every Pareto optimal risk-sharing contract Y, there exists a $\lambda^* > 0$ such that Y has the same distribution as $m^{-1}\left(\lambda^* \delta'\left(T_1(U)\right)\right)$ under P.

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Corollary

If there exists $i \in \{1,2\}$ such that for all $z \in (0,1)$,

$$(\bigstar) \qquad \frac{\widetilde{T}_i''(z)}{\widetilde{T}_i'(z)} > \frac{T_j''(z)}{T_j'(z)},$$

for $\widetilde{T}_i(z) := 1 - T_i(1-z)$, then the risk-sharing contract Y^* is PO if there exists some $\lambda^* > 0$ such that

$$Y^* = m^{-1}\left(\lambda^* \left(\frac{T_2'(1-U)}{T_1'(U)}\right)\right).$$

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Condition $(\bigstar)$ is satisfied for instance when both T_1 and T_2 are concave.

Example with Inverse S-shaped Probability Weighting Functions

- As in Tversky and Kahneman (1992), let the distortion function T_i be given by:

$$T_i(t) = \frac{t^{\gamma_i}}{\left(t^{\gamma_i} + (1-t)^{\gamma_i}\right)^{1/\gamma_i}}, \quad \forall t \in [0,1],$$

  for some $\gamma_i \in (0,1]$.
- It then follows that:

$$\Psi(t) = 1 - \frac{\left(1 - T_1^{-1}(t)\right)^{\gamma_2}}{\left(\left(T_1^{-1}(t)\right)^{\gamma_2} + \left(1 - T_1^{-1}(t)\right)^{\gamma_2}\right)^{1/\gamma_2}}, \quad \forall t \in [0,1].$$

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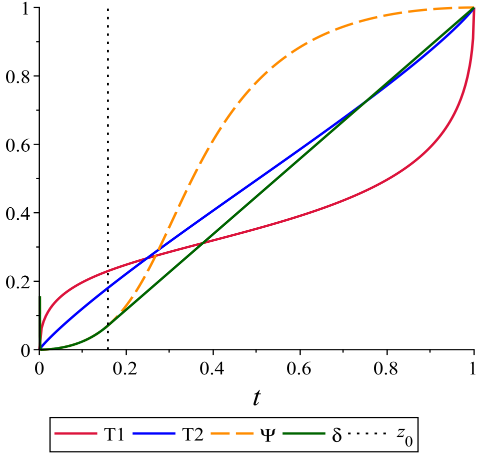
Example, continued

Let $\gamma_1 = 0.5$ and $\gamma_2 = 0.9$. Then:



Example, continued

- Let $u_i(x) = \frac{-\exp(-\beta_i x)}{\beta_i}$, for $x \in \mathbb{R}$ and $\beta_i > 0$.
- $m(x) = \exp((\beta_1 + \beta_2)x)$ for $x \in \mathbb{R}$, and so $m^{-1}(y) = \ln(y)/(\beta_1 + \beta_2)$ for $y > 0$.
- Let $\beta_1 = 0.5$ and $\beta_2 = 0.5$. A risk-sharing contract Y^* is PO if there exists some $\lambda^* > 0$ such that

$$Y^* = m^{-1}\left(\lambda^* \delta'\left(T_1\left(U\right)\right)\right) = \left(\frac{1}{\beta_1 + \beta_2}\right) \ln\left(\lambda^* \delta'\left(T_1\left(U\right)\right)\right)$$
$$= \ln\left(\lambda^*\right) + \ln\left(\delta'\left(T_1\left(U\right)\right)\right).$$

- Thus, the choice of $\lambda^* > 0$ leads to a deterministic side-payment (positive or negative), in addition to the risk-sharing contract $I^*\left(U\right) := \ln\left(\delta'\left(T_1\left(U\right)\right)\right)$.

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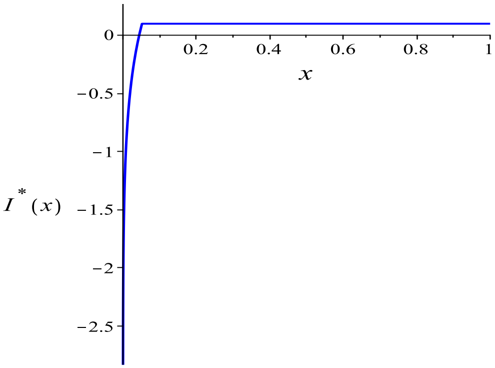

Figure: This graph plots the function I^*, where $I^*(U) := \ln\left(\delta'\left(T_1(U)\right)\right)$ and U is a random variable on (S, Σ, P) with a uniform distribution on $(0, 1)$. Agent 1 receives "large" gains with small probability (*gambling*)

Sunspots

Theorem

The following are equivalent:

(1) $\Psi(t) := \widetilde{T}_2\left(T_1^{-1}(t)\right) \geqslant t$ *for all* $t \in [0,1]$.

(2) *There exists a Pareto optimal no-betting allocation.*

(3) *Any Pareto optimal risk-sharing contract is a no-betting allocation.*

(4) *Every no-betting allocation is Pareto optimal.*

Here, (1) writes as

$$T_1(z) + T_2(1-z) \leqslant 1, \text{ or } T_1(z) - z + T_2(1-z) - (1-z) \leqslant 0, \text{ for all } z \in [0,1].$$

For instance, if for a small $z \in (0,1)$, Agent 1 over-weights good outcomes ($T_1(z) > z$) and Agent 2 under-weights bad outcomes ($T_2(1-z) > 1-z$), there is a desire to shift losses from Agent 1 to Agent 2, and thus random Pareto optimal risk-sharing contracts appear.

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Corollary

If for all $z \in (0,1)$,

$$(\star\star) \qquad \frac{\tilde{T}_i''(z)}{\tilde{T}_i'(z)} \leqslant \frac{T_j''(z)}{T_j'(z)},$$

then $\Psi(t) \geqslant t$ *for all* $t \in [0,1]$.

- Condition $(\star\star)$ holds, for instance, when both T_1 and T_2 are convex.
- Condition $(\star\star)$ holds when both T_1 and T_2 are linear, and thus when both agents are EU maximizers.
- More generally, Condition $(\star\star)$ can be seen as a requirement on the the degree of relative probabilistic risk aversion of the one agent compared to the other one.

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Conclusion

We give an explicit characterization of optimal risk-sharing contracts, in various situations.
In particular, we show that:

(i) Betting is not PO when the two agents are averse to mean-preserving increases in risk (i.e., distortions are convex).

(ii) If the distortions are non-convex, then betting (non-constant) allocations are PO if it does not hold that $\Psi(s) \geqslant s$.

$\Longrightarrow$ Betting or no betting, this thus *only* follows from distortions T_i; *not* on the utilities.

(iii) The set of PO is fully described.

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