BILATERAL RISK SHARING WITH NO AGGREGATE UNCERTAINTY UNDER RANK-DEPENDENT UTILITY

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THIS DRAFT: APRIL 27, 2020

Abstract. This paper studies bilateral risk-sharing with no aggregate uncertainty, when agents maximize rank-dependent utilities. We characterize the structure of Pareto optimal risk-sharing contracts in full generality. We then derive a necessary and sufficient condition for Pareto optima to be no-betting allocations (i.e., deterministic allocations), thereby answering the question of when sunspots do not exist in this economy. This condition depends only on the probability weighting functions of the two agents, and not on their (concave) utility functions.

1. Introduction

Starting from a situation of no aggregate uncertainty, when will two economic agents want to bet on an extrinsic event, thereby introducing uncertainty in the economy? This paper examines conditions on the probability weighting functions of two Rank-Dependent Utility (RDU) maximizing agents under which engaging in uncertainty-generating trade (i.e., betting) is Pareto-improving.

As a special case of risk-sharing in the situation of no aggregate uncertainty, Cass and Shell [5] propose the concept of *sunspot equilibria*. A "sunspot" represents an extrinsic random variable; and a sunspot occurs when there exists a Pareto optimal risk-sharing contract that is

Key Words and Phrases: Risk Sharing, Pareto Optimality, Sunspots, Rank-Dependent Utility.

JEL Classification: C02, D86, G22.

2010 Mathematics Subject Classification: 91B30.

We are grateful to Jean-Marc Tallon and Sebastian Ebert for comments and suggestions. Mario Ghossoub acknowledges financial support from the Natural Sciences and Engineering Research Council of Canada (NSERC Grant No. 2018-03961).

not constant across states of the world, even when starting from a situation of deterministic initial endowments. In other words, if sunspots do not occur, then every Pareto optimal risk-sharing contract is a no-betting allocation. We provide conditions under which sunspots exist for bilateral risk-sharing markets with RDU preferences, as proposed by Quiggin [19, 20] and Schmeidler's [22] more general Choquet Expected Utility (CEU) model. We do not attempt to argue that sunspots are realistic or empirically observed. Indeed, on the one hand, if sunspots do not matter then betting or gambling cannot happen in *any* exchange market. On the other hand, if sunspots matter then we would expect that many agents bet on events, as long as their probability weighting functions satisfy a certain condition. In other words, we would expect a significant amount of betting to occur on extrinsic events.

In Expected-Utility (EU) theory, it is well-known that sunspots do not exist when the risk-averse agents have the same probabilistic beliefs. This observation is generalized by Billot et al. [2, 3] to the case of agents with Maxmin Expected Utility (MEU) preferences, as proposed by Gilboa and Schmeidler [14]. They show that, for risk-averse agents, sunspots do not exist if and only if the agents share a common prior. More recently, Dominiak et al. [11] and Ghirardato and Siniscalchi [13] extend the characterization of Billot et al. [2, 3] to a class of non-convex preferences. The focus of this paper is on RDU agents characterized by generic probability weighting functions and concave utility functions. In line with the literature on sunspot equilibria, we focus on the Pareto optimal risk-sharing contracts in markets with no aggregate uncertainty. We provide a condition under which Pareto optimal contracts are no-betting contracts, i.e., constant across states of the world. This result generalizes the results of Tallon [23] and De Castro and Chateauneuf [10]. Tallon [23] shows that sunspots exist when the probability weighting function is concave (hence yielding a subadditive capacity), and De Castro and Chateauneuf [10] show that it is Pareto optimal not to trade under a specific condition on the capacities of the CEU agents involved. While we focus on RDU preferences (and hence on probability weighting functions instead of capacities), we show that this condition is also necessary and sufficient for non-existence of sunspots. For a general class of convex preferences, Rigotti et al. [21] show that there are only deterministic Pareto optimal risk-sharing contracts if there is no aggregate market uncertainty. We characterize this situation as well, but for RDU preferences that are allowed to be non-convex, which means in our case that we allow probability weighting functions to be non-convex. A convex probability weighting function implies that the RDU agent is averse to mean-preserving spreads (see, e.g., Chew et al. [8]). Quiggin [20] and Tversky and Kahneman [25] suggest that an inverse-S shaped probability weighting function is more plausible to describe human behavior, which implies that individuals overweight extreme (good and bad) events. We illustrate the effect of inverse S-shaped probability weighting functions in an example.

Risk sharing received considerable attention in markets with aggregate market uncertainty, in which trading a state-contingent payoff is interpreted as hedging rather than betting. For instance, when the agents are endowed with RDU or CEU preferences, Pareto optimal risk-sharing is studied by Chateauneuf et al. [7], Tsanakas and Christofides [24], Carlier and Dana [4], Chakravarty and Kelsey [6], Xia and Zhou [26], and Jin et al. [16]. Chateauneuf et al. [7], Tsanakas and Christofides [24], Chakravarty and Kelsey [6], and Carlier and Dana [4] all assume that the probability weighting functions are convex, and thus individuals are averse to mean-preserving spreads. Xia and Zhou [26] assume that all agents use the same probability weighting function. Jin et al. [16] show that Pareto optimal risk-sharing contracts exist under technical conditions that require an atomless state-price density, hence implying the existence of aggregate market uncertainty. Our main contribution to the literature is to allow for non-convex probability weighting functions (such as an inverse-S shaped function) in the case of no aggregate market uncertainty.

This paper is set out as follows. In Section 2, we introduce the RDU preferences of the agents. Our general risk-sharing result is presented in Section 3, as well as an example that illustrates the case of inverse S-shaped probability weighting functions. This general result is then used in Section 4 to provide a necessary and sufficient condition under which there only exist Pareto optimal risk-sharing contracts that are no-betting allocations. Finally, Section 5 concludes. All proofs are presented in the Appendices.

2. Preferences of the Agents

Let (S, Σ, P) be a non-atomic probability space, and let $B(\Sigma)$ be the vector space of all bounded, $\mathbb{R}$-valued, and Σ-measurable functions on (S, Σ). For any $Z \in B(\Sigma)$, let F_Z and F_Z^{-1} denote the cumulative distribution function and quantile function of Z, respectively, defined by:

$$F_Z(t) := P\left(\{s \in S : Z(s) \leqslant t\}\right), \ \forall t \geqslant 0;$$

and

$$F_Z^{-1}(t) := \inf\left\{z \in \mathbb{R} \;\middle|\; F_Z(z) \geqslant t\right\}, \ \forall t \in [0,1].$$

We assume that there are two agents and that there is no aggregate uncertainty. The agents seek a risk-sharing arrangement $(X_1, X_2) \in B(\Sigma) \times B(\Sigma)$ such that $X_1 + X_2 = 0$, which is interpreted as betting if this arrangement is not deterministic. The utility function of agent $i \in \{1, 2\}$ takes the form

$$\int \hat{u}_i\Big(W_i - X_i\Big) \ dT_i \circ P, \tag{2.1}$$

where $W_i \in \mathbb{R}$ and $\hat{u}_i$ are respectively the initial wealth and utility function of agent $i \in \{1, 2\}$, and integration is in the sense of Choquet as defined below.

Definition 2.1. For any $Z \in B(\Sigma)$ and $i \in \{1, 2\}$, the Choquet integral of $\hat{u}_i(Z)$ with respect to $T_i \circ P$ is defined as

$$\begin{aligned} \int \hat{u}_i(Z) \ dT_i \circ P := \int_0^{+\infty} T_i\Big(P\Big(\{s \in S : \hat{u}_i(Z(s)) > t\}\Big)\Big) \ dt \\ + \int_{-\infty}^0 \Big[T_i\Big(P\Big(\{s \in S : \hat{u}_i(Z(s)) > t\}\Big)\Big) - 1\Big] \ dt. \end{aligned} \tag{2.2}$$

Assumption 2.2. *The utility functions $\hat{u}_i$ are increasing,*[1] *strictly concave, continuously differentiable, and satisfy the Inada conditions* $\lim_{x\to-\infty} \hat{u}_i'(x) = +\infty$ *and* $\lim_{x\to+\infty} \hat{u}_i'(x) = 0$.

[1] Throughout this paper, we mean "increasing" in the strictly increasing sense. We use the terminology "non-decreasing" to mean weakly increasing. We use the same convention for decreasing functions.

Moreover, the probability weighting functions $T_i : [0,1] \to [0,1]$ are such that $T_i(0) = 0$, $T_i(1) = 1$, and functions T_i are absolutely continuous and increasing.

If the probability weighting function T_i is convex and $\hat{u}_i$ is concave, Chew et al. [8] show that agent i is averse to mean-preserving spreads. A RDU preference representation is a special case of CEU, in which the agent's non-additive measure (sometimes called a capacity) υ is a distortion of a probability measure ($\upsilon = T \circ \mu$, for some probability measure μ). In this case, convexity (resp. concavity) of the probability weighting function T yields convexity (resp. concavity) of the capacity ν. In CEU, a convex capacity reflects ambiguity-aversion, while a concave capacity reflects ambiguity-seeking behavior.

3. Pareto Optimal Risk-Sharing Contracts

For $i \in \{1, 2\}$, define the function u_i by $u_i(x) := \hat{u}_i(W_i + x)$ for all $x \in \mathbb{R}$. Then u_i satisfies Assumption 2.2 whenever $\hat{u}_i$ does. Letting $Y := X_1$, we can then reformulate the risk-sharing problem as that of finding a Pareto optimal $Y \in B(\Sigma)$, when Agent 1 maximizes

$$\int u_1\Big(- Y\Big) \; dT_1 \circ P,$$

and Agent 2 maximizes

$$\int u_2\Big(Y\Big) \; dT_2 \circ P.$$

Note that a positive realization of Y is interpreted as a transfer of wealth from Agent 1 to Agent 2, while a negative realization of Y indicates a transfer of wealth from Agent 2 to Agent 1. We will refer to Y as a risk-sharing contract. We next define the concept of Pareto optimality in this context.

Definition 3.1. The risk-sharing contract $Y \in B(\Sigma)$ is Pareto optimal if there does not exist any other risk-sharing contract $Z \in B(\Sigma)$ such that

$$\int u_1(-Y) \; dT_1 \circ P \leqslant \int u_1(-Z) \; dT_1 \circ P, \text{ and } \int u_2(Y) \; dT_2 \circ P \leqslant \int u_2(Z) \; dT_2 \circ P,$$

with at least one strict inequality.

Next, we characterize all Pareto optimal risk-sharing contracts in full generality. First recall that the convex envelope of a function is the greatest convex function that is point-wise dominated by that function[2]

Theorem 3.2. *If Assumption 2.2 holds, then a risk-sharing contract Y^* is Pareto optimal if there exists some $\lambda^* > 0$ such that*

$$Y^* = m^{-1}\left(\lambda^* \delta'\Big(T_1(U)\Big)\right),$$

where:

[2]For a real-valued function f on a non-empty convex subset of $\mathbb{R}$ containing the interval $[0,1]$, the convex envelope of f on the interval $[0,1]$ is defined as the greatest convex function g on $[0,1]$ such that $g(x) \leqslant f(x)$, for each $x \in [0,1]$.

- *U is a random variable on (S, Σ, P) with a uniform distribution on $(0,1)$;*
- $m(x) := \frac{u_1'(-x)}{u_2'(x)}$*, for all* $x \geqslant 0$;
- *δ is the convex envelope on $[0,1]$ of the function $\Psi : [0,1] \to \mathbb{R}$ defined by*

$$\Psi(t) := \widetilde{T}_2\left(T_1^{-1}(t)\right); \tag{3.1}$$

- $\widetilde{T}_2(t) = 1 - T_2(1-t)$*, for each* $t \in [0,1]$.

Moreover, for every Pareto optimal risk-sharing contract Y, there exists a $\lambda^* > 0$ such that Y has the same distribution as $m^{-1}\left(\lambda^* \delta'\left(T_1(U)\right)\right)$ under P.

By virtue of Theorem 3.2, varying $\lambda^* > 0$ traces the entire Pareto frontier. Potential uncertainty is generated by taking any uniform random variable U on (S, Σ, P). To generate a Pareto optimal risk-sharing contract Y, a non-decreasing transformation is applied to U.

Next, we show that Theorem 3.2 can be simplified under a specific condition on the probability weighting functions.

Theorem 3.3. *Let $\widetilde{T}_i$ be the conjugate of T_i, defined by $\widetilde{T}_i(z) := 1 - T_i(1-z)$. If Assumption 2.2 holds and there exists $i \in \{1,2\}$ such that for all $z \in (0,1)$,*

$$\frac{\widetilde{T}_i''(z)}{\widetilde{T}_i'(z)} > \frac{T_j''(z)}{T_j'(z)}, \tag{$\bigstar$}$$

where $j = 3 - i$, then the risk-sharing contract Y^* is Pareto optimal if there exists some $\lambda^* > 0$ such that

$$Y^* = m^{-1}\left(\lambda^* \left(\frac{T_2'(1-U)}{T_1'(U)}\right)\right),$$

where:

- *U is a random variable on (S, Σ, P) with a uniform distribution on $(0,1)$;*
- $m(x) := \frac{u_1'(-x)}{u_2'(x)}$*, for all* $x \geqslant 0$.

Moreover, for every Pareto optimal risk-sharing contract Y, there exists a $\lambda^* > 0$ such that Y has the same distribution as $m^{-1}\left(\lambda^* \left(\frac{T_2'(1-U)}{T_1'(U)}\right)\right)$ under P.

Condition ($\bigstar$) is satisfied for instance when both T_1 and T_2 are concave. In that case, Theorem 3.3 states that the Pareto optimal risk-sharing contracts are not constant across states. Moreover, this theorem implies that, unlike in EU theory, common probabilistic beliefs might still lead to a risk-sharing situation in which betting is Pareto-improving, depending on the shape of the probability weighting functions. In the next section, we study conditions under which all Pareto optimal risk-sharing contracts are deterministic. However, we first examine the special case of inverse S-shaped probability weighting functions.

Example 3.4 (Inverse S-shaped Probability Weighting Functions). As an illustration, we consider the case of inverse S-shaped probability weighting functions, as in Tversky and Kahneman [25]. Specifically, we suppose that for $i \in \{1, 2\}$, the distortion function T_i is given by

$$T_i(t) = \frac{t^{\gamma_i}}{\left(t^{\gamma_i} + (1-t)^{\gamma_i}\right)^{1/\gamma_i}}, \ \forall t \in [0,1], \tag{3.2}$$

for some $\gamma_i \in (0, 1]$. It then follows that the function $\Psi : [0, 1] \to \mathbb{R}$ defined in eq. (3.1) is given by

$$\Psi(t) = 1 - T_2\left(1 - T_1^{-1}(t)\right) = 1 - \frac{\left(1 - T_1^{-1}(t)\right)^{\gamma_2}}{\left(\left(T_1^{-1}(t)\right)^{\gamma_2} + \left(1 - T_1^{-1}(t)\right)^{\gamma_2}\right)^{1/\gamma_2}}, \ \forall t \in [0,1].$$

For values $\gamma_1 = 0.5$ and $\gamma_2 = 0.9$, it is easily verified that there exists $t_0 \in [0, 1]$ such that Ψ is convex on the interval $[0, t_0]$ and concave on the interval $[t_0, 1]$. Let δ be the convex envelope of Ψ on the interval $[0, 1]$. Then $\Psi(0) = \delta(0) = 0$ and $\Psi(1) = \delta(1) = 1$. Moreover, since δ is affine on the set $\{t \in [0, 1] : \delta(t) < \Psi(t)\}$, there exists some $z_0 \in (0, t_0)$ such that δ is given by

$$\delta(t) = \begin{cases} \Psi(t) & \text{if } t \leqslant z_0; \\ \Psi(z_0) + \left(\frac{1-\Psi(z_0)}{1-z_0}\right)(t - z_0) & \text{if } t \geqslant z_0. \end{cases}$$

Note that δ is continuously differentiable by continuity of Ψ, and moreover it holds that $\delta'(z) = \frac{1-\Psi(z_0)}{1-z_0}$ for all $z \geqslant z_0$. Numerical computation gives $z_0 \approx 0.1578$. Figure 1 plots the graph of the functions T_1, T_2, Ψ, and δ.

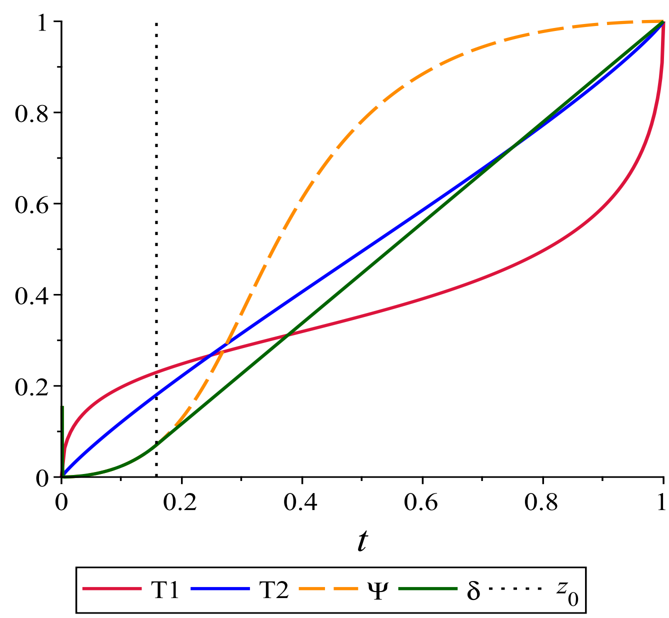

FIGURE 1. This graph plots the function T_1 (solid red line), the function T_2 (solid blue line), the function Ψ (dashed orange line), and the convex envelope δ of Ψ (solid dark green line). The dotted vertical black line is the graph of the function $g(t) := z_0$, for $t \in [0, 1]$.

Suppose that, for $i \in \{1,2\}$, $u_i(x) = \frac{-\exp(-\beta_i x)}{\beta_i}$, for $x \in \mathbb{R}$ and $\beta_i > 0$. Moreover, $m(x) = \exp((\beta_1 + \beta_2)x)$ for $x \in \mathbb{R}$, and so $m^{-1}(y) = \ln(y)/(\beta_1 + \beta_2)$ for $y > 0$. In general, m^{-1} is an increasing function with $\lim_{y\to 0} m^{-1}(y) = -\infty$ and $\lim_{y\to\infty} m^{-1}(y) = \infty$. We let $\beta_1 = 0.5$ and $\beta_2 = 0.5$, so that $m^{-1}(y) = \ln(y)$, for $y > 0$. Theorem 3.2 then implies that a risk-sharing contract Y^* is Pareto optimal if there exists some $\lambda^* > 0$ such that

$$
\begin{aligned}
Y^* &= m^{-1}\left(\lambda^* \delta'\Big(T_1\left(U\right)\Big)\right) = \left(\frac{1}{\beta_1 + \beta_2}\right) \ln\left(\lambda^* \delta'\Big(T_1\left(U\right)\Big)\right) \\
&= \ln\left(\lambda^*\right) + \ln\left(\delta'\Big(T_1\left(U\right)\Big)\right).
\end{aligned}
\tag{3.3}
$$

Thus, the choice of $\lambda^* > 0$ leads to a deterministic side-payment (positive or negative), in addition to the risk-sharing contract $I^*\left(U\right) := \ln\left(\delta'\Big(T_1\left(U\right)\Big)\right)$. By varying side-payments in $\mathbb{R}$, we span the set of Pareto optimal risk-sharing contracts.

Moreover, for $i \in \{1,2\}$, the shape of this risk-sharing contract depends on β_i only via $\beta_1 + \beta_2$. For the same value of $\beta_1 + \beta_2$, the differences in risk-aversion parameters β_i may only be reflected in the value of the side-payment (see eq. (3.3)). Note that on the interval $[T_1^{-1}(z_0), 1] \approx [0.052, 1]$ of realizations of U, the realization of Y^* is the same. On the interval $[0, T_1^{-1}(z_0)] = [0, 0.052]$ of realizations of U, the risk-sharing contract Y^* is strictly increasing in U. For $\lambda^* = 1$, it holds that $\ln(\lambda^*) = 0$, and thus the side-payment is equal to zero and hence $Y^* = I^*(U)$. We display in Figure 2 the function I^*. Recall that a positive realization of Y^* is interpreted as a transfer of wealth from Agent 1 to Agent 2, while a negative realization of Y indicates a transfer of wealth from Agent 2 to Agent 1. A small value of U yields a "good" realization for Agent 1 and a "bad" realization for Agent 2.

We now look at certainty-equivalents for the case of $\lambda^* = 1$. The certainty equivalents of a risk-sharing contract Y^* for Agent 1 and Agent 2 are defined, respectively, as

$$
CEQ_1 := u_1^{-1}\left(\int u_1\left(-Y^*\right) dT_1 \circ P\right) \quad \text{and} \quad CEQ_2 := u_2^{-1}\left(\int u_2\left(Y^*\right) dT_2 \circ P\right).
$$

Numerical computation then yields $CEQ_1 \approx 3.32\%$ and $CEQ_2 \approx 1.69\%$. Since both are positive, the risk-sharing contract displayed in Figure 2 yields a higher utility for both agents than in the absence of risk sharing (that is, when $Y^* \equiv 0$). By varying $\lambda^* > 0$, the aggregate welfare gain remain the same, i.e., $CEQ_1 + CEQ_2 = 5.01\%$. Any allocation (CEQ_1, CEQ_2) such that $CEQ_1 + CEQ_2 = 5.01\%$ can be obtained by varying $\lambda^* > 0$, and thus by varying the deterministic side-payments.

In Example 3.4 above, we studied a situation where the Pareto optimal risk-sharing contracts are constant for realizations of U in the interval $[0.052, 1]$. In the next section, we examine conditions under which the Pareto optimal risk-sharing contracts are deterministic.

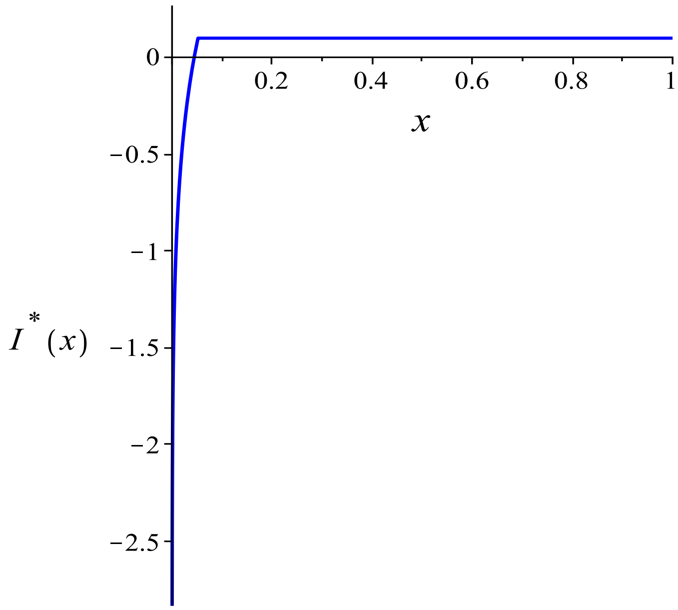

FIGURE 2. This graph plots the function I^*, where $I^*(U) := \ln\Big(\delta'\Big(T_1(U)\Big)\Big)$ and U is a random variable on (S, Σ, P) with a uniform distribution on $(0,1)$.

4. No-Betting Allocations

In this section, our focus is on situations in which Pareto optimal risk-sharing contracts are no-betting allocations. A no-betting allocation occurs when Y has the same distribution as a deterministic random variable. That is, a no-betting allocation is one that is deterministic almost surely. The following result provides necessary and sufficient conditions under which there exist Pareto optimal allocations that are no-betting allocations.

Theorem 4.1. *Let the function* $\Psi : [0,1] \to \mathbb{R}$ *be defined as in eq.* (3.1)*. If Assumption 2.2 holds, then the following are equivalent:*

(1) $\Psi(t) \geqslant t$ *for all* $t \in [0,1]$.

(2) *There exists a Pareto optimal no-betting allocation.*

(3) *Any Pareto optimal risk-sharing contract is a no-betting allocation.*

(4) *Every no-betting allocation is Pareto optimal.*

Following Cass and Shell [5], sunspot equilibria are studied in the case of deterministic endowments, which is a special case of the setting studied in this paper. Thus, Theorem 4.1 also characterizes situations in which sunspots exist, i.e., when Pareto optimal risk-sharing contracts depend on an extrinsic random variable. In contrast to MEU preferences, sunspots can exist with RDU preferences when probabilistic beliefs are given and common (equal to $\mathbb{P}$) and the source of trading is differential risk attitudes. In fact, it may be Pareto optimal to trade when the probability weighting functions are not convex.

As long as the probability weighting functions are such that $\Psi(t) \geqslant t$ for all $t \in [0,1]$, Pareto optimal risk-sharing contracts are no-betting allocations. Otherwise, betting is Pareto optimal. Note that $\Psi(t) \geqslant t$ is equivalent to $T_1^{-1}(t) \geqslant \widetilde{T}_2^{-1}(t)$ for all $t \in [0,1]$. If we substitute $t = T_1(z)$, this rewrites as $T_1(z) \leqslant \widetilde{T}_2(z)$, or equivalently

$$T_1(z) + T_2(1-z) \leqslant 1, \text{for all } z \in [0,1].$$

We interpret this as an elevation property of the probability weighting functions of the two agents jointly, since this writes as

$$T_1(z) - z + T_2(1-z) - (1-z) \leqslant 0, \text{ for all } z \in [0,1]. \tag{4.1}$$

For instance, if for a small $z \in (0,1)$, Agent 1 over-weights good outcomes ($T_1(z) > z$) and Agent 2 under-weights bad outcomes ($T_2(1-z) > 1-z$), there is a desire to shift losses from Agent 1 to Agent 2, and thus random Pareto optimal risk-sharing contracts appear. Moreover, this also holds vice versa for large $z \in (0,1)$.

When both agents use the same probability weighting function, this elevation property is associated with *pessimism* or *source preference* (see Abdellaoui et al. [1]). It is called pessimism as it has the implication that the decision weight of the worst outcome is larger than the probability of obtaining that outcome. If $T_1 = T_2$, this property is also called the *subcertainty* effect, which is a basic property imposed in prospect theory (Kahneman and Tversky [17]). For instance, Condition (4.1) holds true when $T_1(t) = T_2(t) = t^\gamma/((t^\gamma + (1-t)^\gamma)^{1/\gamma})$ for $\gamma \in (0.279, 1]$, which is inverse-S shaped (Tversky and Kahneman [25]).

De Castro and Chateauneuf [10] already show that under a condition that is similar to our Condition (4.1), the no-betting allocation $Y \overset{d}{=} 0$ is Pareto optimal for CEU agents. Their condition coincides with our Condition (4.1) in the special case of RDU preferences, that is, when the capacities of the CEU agents are distortions of a baseline probability measure. Theorem 4.1 shows that under Condition (4.1) *all* Pareto optima are no-betting allocations. Moreover, Theorem 4.1 shows that Condition (4.1) is also necessary for Pareto optimality of no-betting allocations.

We next study a more specific condition that implies $\Psi(t) \geqslant t$ for all $t \in [0,1]$.

Proposition 4.2. *If Assumption 2.2 holds and, for all $z \in (0,1)$,*

$$(\star\star) \qquad \frac{\widetilde{T}_i''(z)}{\widetilde{T}_i'(z)} \leqslant \frac{T_j''(z)}{T_j'(z)},$$

then $\Psi(t) \geqslant t$ for all $t \in [0,1]$.

Condition ($\star\star$) holds, for instance, when both T_1 and T_2 are convex. This implies that both agents are averse to mean-preserving spreads. Also, Condition ($\star\star$) holds when both T_1 and T_2 are linear, and thus when both agents are EU maximizers. More generally, Condition ($\star\star$) can be seen as a requirement on the the degree of relative probabilistic risk aversion of the one agent compared to the other one.

5. Conclusion

This paper studies bilateral Pareto optimal risk-sharing with rank-dependent utilities in an economy that has no aggregate uncertainty. This paper's contribution is to characterize the set of all Pareto optimal contracts. Moreover, we identify a condition that characterizes situations in which sunspots matter, i.e., when the two agents find it mutually beneficial to take bets, thereby introducing uncertainty in the economy. This condition only depends on the probability weighting functions.

It is important to note that our results depend critically on the assumption of no aggregate uncertainty. With aggregate uncertainty, one only needs to search for risk-sharing contracts that are comonotonic with the aggregate risk, since Pareto optimal risk-sharing contracts are necessarily comonotonic (see, e.g., Föllmer and Schied [12] and Jin et al. [16]). This may not be possible to find, and hence Pareto optimal risk-sharing contracts may not exist. We leave this topic open for future research.

Appendix A. Proof of Theorem 3.2

A.1. **An Auxiliary Problem.** First, we introduce the following auxiliary problem:

Problem A.1 ($\widehat{P}_{V_0}$)**.** *For $V_0 \in \mathbb{R}$,*

$$\left(\widehat{\mathcal{P}}_{V_0}\right) \qquad \sup_{Y \in B(\Sigma)} \left\{ \int u_1\left(-Y\right)\ dT_1 \circ P : \int u_2\left(Y\right)\ dT_2 \circ P \geqslant V_0 \right\}.$$

Similar to the ε-constraint method for multi-objective optimization (see, e.g., Cohon [9] and Miettinen [18]), our aim is to span the set of Pareto optimal contracts by allowing $V_0 \in \mathbb{R}$ to be flexible. The following result characterizes the relationship between Pareto optimal risk-sharing contracts and the solutions to Problem $\left(\widehat{\mathcal{P}}_{V_0}\right)$, for some $V_0 \in \mathbb{R}$.

Lemma A.2.

(i) *If the risk-sharing contract $Y^* \in B\left(\Sigma\right)$ is Pareto optimal, then it solves Problem $\left(\widehat{\mathcal{P}}_{V_0}\right)$ with $V_0 := \int u_2\left(Y^*\right)\ dT_2 \circ P$;*

(ii) *for a given $V_0 \in \mathbb{R}$, any solution to Problem $\left(\widehat{\mathcal{P}}_{V_0}\right)$ is Pareto optimal;*

(iii) *if $Y^* \in B\left(\Sigma\right)$ solves Problem $\left(\widehat{\mathcal{P}}_{V_0}\right)$ for a given $V_0 \in \mathbb{R}$, then $\int u_2\left(Y^*\right)\ dT_2 \circ P = V_0$.*

Proof.

(i) Suppose first that Y^* is Pareto optimal but that Y^* does not solve Problem $\left(\widehat{\mathcal{P}}_{V_0}\right)$ with $V_0 = \int u_2\left(Y^*\right)\ dT_2 \circ P$. Then, there exist $Y \in B(\Sigma)$ that solves Problem $\left(\widehat{\mathcal{P}}_{V_0}\right)$ with V_0. Therefore,

$$\int u_1\left(-Y\right)\ dT_1 \circ P > \int u_1\left(-Y^*\right)\ dT_1 \circ P \text{ and } \int u_2\left(Y\right)\ dT_2 \circ P \geqslant V_0 = \int u_2\left(Y^*\right)\ dT_2 \circ P.$$

Thus, Y is a Pareto improvement over Y^*, contradicting the Pareto optimality of Y^*.

(ii) We continue with the second statement. Fix $V_0 \in \mathbb{R}$, and let Y^* be a solution to Problem $\left(\widehat{\mathcal{P}}_{V_0}\right)$. Suppose that Y^* is not Pareto optimal, so that there exists $Y \in B(\Sigma)$ such that

$$\int u_1\left(-Y\right)\ dT_1 \circ P \geqslant \int u_1\left(-Y^*\right)\ dT_1 \circ P, \text{ and } \int u_2\left(Y\right)\ dT_2 \circ P \geqslant \int u_2\left(Y^*\right)\ dT_2 \circ P,$$

with at least one strict inequality. Then, in particular, Y is feasible for Problem $\left(\widehat{\mathcal{P}}_{V_0}\right)$. If $\int u_1\left(-Y\right)\ dT_1 \circ P > \int u_1\left(-Y^*\right)\ dT_1 \circ P$, this contradicts the optimality of Y^* for

Problem $\left(\widehat{\mathcal{P}}_{V_0}\right)$. Therefore, $\int u_1(-Y)\ dT_1 \circ P = \int u_1(-Y^*)\ dT_1 \circ P$. Assume then that $\int u_2(Y)\ dT_2 \circ P > \int u_2(Y^*)\ dT_2 \circ P$, and let ε be such that $\int u_2(Y-\varepsilon)\ dT_2 \circ P = \int u_2(Y^*)\ dT_2 \circ P$. Then $\varepsilon > 0$, by strict monotonicity of u_2. Letting $\bar{Y} := Y - \varepsilon$ it follows from the feasibility of Y^* for Problem $\left(\widehat{\mathcal{P}}_{V_0}\right)$ that $\bar{Y}$ is feasible for Problem $\left(\widehat{\mathcal{P}}_{V_0}\right)$. Moreover, since u_1 is increasing, it follows that

$$\int u_1\left(-\bar{Y}\right) dT_1 \circ P = \int u_1(-Y+\varepsilon)\, dT_1 \circ P > \int u_1(-Y)\, dT_1 \circ P = \int u_1(-Y^*)\, dT_1 \circ P,$$

which contradicts the optimality of Y^* for Problem $\left(\widehat{\mathcal{P}}_{V_0}\right)$. Hence Y^* is Pareto optimal.

(iii) We conclude with the proof of the third statement. Fix $V_0 \in \mathbb{R}$, and let Y^* be a solution to Problem $\left(\widehat{\mathcal{P}}_{V_0}\right)$. Let ε be such that $\int u_2(Y^*-\varepsilon)\, dT_2 \circ P = V_0$. Then $\varepsilon > 0$. Letting $\bar{Y} := Y^* - \varepsilon$, it follows that $\bar{Y}$ is feasible for Problem $\left(\widehat{\mathcal{P}}_{V_0}\right)$. Moreover, since u_1 is increasing,

$$\int u_1\left(-\bar{Y}\right)\ dT_1 \circ P = \int u_1(-Y^*+\varepsilon)\ dT_1 \circ P > \int u_1(-Y^*)\ dT_1 \circ P,$$

contradicting the optimality of Y^* for Problem $\left(\widehat{\mathcal{P}}_{V_0}\right)$. Therefore, $\int u_2(Y^*)\ dT_2 \circ P = V_0$.

□

A.2. **Solving the Auxiliary Problem.** We now provide a solution of Problem $\left(\widehat{\mathcal{P}}_{V_0}\right)$.

Theorem A.3. *For a given $V_0 \in \mathbb{R}$, the risk-sharing contract*

$$Y^* := m^{-1}\left(\lambda^* \delta'\left(T_1(U)\right)\right)$$

is optimal for Problem $\left(\widehat{\mathcal{P}}_{V_0}\right)$, where:

- *U is a random variable on (S, Σ, P) with a uniform distribution on $(0,1)$;*
- *$m(x) := \frac{u_1'(-x)}{u_2'(x)}$, for all $x \geqslant 0$;*
- *δ is the convex envelope on $[0,1]$ of $\Psi(t) := \widetilde{T}_2\left(T_1^{-1}(t)\right)$;*
- *$\widetilde{T}_2(t) = 1 - T_2(1-t)$; and,*
- *$\lambda^* > 0$ is chosen such that $\int u_2(Y^*)\, dT_2 \circ P = V_0$.*

We now turn to the proof of Theorem A.3. First note that, since the space (S, Σ, P) is non-atomic by assumption, there exists a random variable U on (S, Σ, P) with a uniform distribution on $(0,1)$ (e.g., [12, Proposition A.31]). Consider the following problem.

Problem A.4.

$$\sup_{f \in \mathcal{Q}^*} \left\{ \int_0^1 u_1\big(-f(t)\big) T_1'(t)\, dt : \int_0^1 u_2\big(f(t)\big) T_2'(1-t)\, dt \geqslant V_0 \right\},$$

where $\mathcal{Q}^*$ is the collection of all quantile functions, i.e.,

$$\mathcal{Q}^* := \Big\{ f : (0,1) \to \mathbb{R} \,\Big|\, f \text{ is non-decreasing and left-continuous} \Big\}. \tag{A.1}$$

We obtain the following result.

Lemma A.5. *If f^* is optimal for Problem A.4, then $Y^* := f^*(U)$ is optimal for Problem $\left(\widehat{\mathcal{P}}_{V_0}\right)$.*

Proof. Let f^* be optimal for Problem A.4 and $Y^* = f^*(U)$. Then, since $f^* \in \mathcal{Q}^*$, it follows that $F_{Y^*}^{-1} = f^*$. Moreover, using the variable $s = F_{u_2(Y^*)}(t)$ yields

$$\begin{aligned}
\int u_2(Y^*) dT_2 \circ P &= \int_0^{+\infty} T_2\left(1 - F_{u_2(Y^*)}(t)\right)\, dt + \int_{-\infty}^0 \left[T_2\left(1 - F_{u_2(Y^*)}(t)\right) - 1\right]\, dt \\
&= \int_{F_{u_2(Y^*)}(0)}^1 T_2(1-s)\, dF_{u_2(Y^*)}^{-1}(s) + \int_0^{F_{u_2(Y^*)}(0)} \left[T_2(1-s) - 1\right]\, dF_{u_2(Y^*)}^{-1}(s) \\
&= \int_{F_{u_2(Y^*)}(0)}^1 \int_0^{1-s} T_2'(z)\, dz\, dF_{u_2(Y^*)}^{-1}(s) \\
&\quad + \int_0^{F_{u_2(Y^*)}(0)} \int_0^{1-s} T_2'(z)\, dz\, dF_{u_2(Y^*)}^{-1}(s) - \int_0^{F_{u_2(Y^*)}(0)} dF_{u_2(Y^*)}^{-1}(s) \\
&= \int_0^{1-F_{u_2(Y^*)}(0)} T_2'(z) \int_{F_{u_2(Y^*)}(0)}^{1-z} dF_{u_2(Y^*)}^{-1}(s)\, dz - \int_0^{F_{u_2(Y^*)}(0)} dF_{u_2(Y^*)}^{-1}(s) \\
&\quad + \int_0^1 T_2'(z) \int_0^{\min\left[1-z, F_{u_2(Y^*)}(0)\right]} dF_{u_2(Y^*)}^{-1}(s)\, dz \\
&= \int_0^{1-F_{u_2(Y^*)}(0)} T_2'(z)\, F_{u_2(Y^*)}^{-1}(1-z)\, dz + F_{u_2(Y^*)}^{-1}(0) \\
&\quad + \int_0^{1-F_{u_2(Y^*)}(0)} T_2'(z) \int_0^{F_{u_2(Y^*)}(0)} dF_{u_2(Y^*)}^{-1}(s)\, dz \\
&\qquad + \int_{1-F_{u_2(Y^*)}(0)}^1 T_2'(z) \int_0^{1-z} dF_{u_2(Y^*)}^{-1}(s)\, dz
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^{1-F_{u_2(Y^*)}(0)} T_2'(z)\, F^{-1}_{u_2(Y^*)}(1-z)\ dz + F^{-1}_{u_2(Y^*)}(0) \\
&\qquad - F^{-1}_{u_2(Y^*)}(0) \int_0^{1-F_{u_2(Y^*)}(0)} T_2'(z)\ dz + \int_{1-F_{u_2(Y^*)}(0)}^1 T_2'(z)\, F^{-1}_{u_2(Y^*)}(1-z)\ dz \\
&\qquad - F^{-1}_{u_2(Y^*)}(0) \int_{1-F_{u_2(Y^*)}(0)}^1 T_2'(z)\ dz \\
&= \int_0^{1-F_{u_2(Y^*)}(0)} T_2'(z)\, F^{-1}_{u_2(Y^*)}(1-z)\ dz + \int_{1-F_{u_2(Y^*)}(0)}^1 T_2'(z)\, F^{-1}_{u_2(Y^*)}(1-z)\ dz \\
&= \int_0^1 T_2'(z)\, F^{-1}_{u_2(Y^*)}(1-z)\ dz = \int_0^1 F^{-1}_{u_2(Y^*)}(t)\, T_2'(1-t)\ dt \\
&= \int F^{-1}_{u_2(Y^*)}(U)\, T_2'(1-U)\, dP = \int u_2\left(F^{-1}_{Y^*}\right)(U)\, T_2'(1-U)\, dP \\
&= \int u_2\left(f^*(U)\right) T_2'(1-U)\, dP = \int_0^1 u_2\left(f^*(t)\right) T_2'(1-t)\, dt \geqslant V_0,
\end{aligned}
$$

where the equality $F^{-1}_{u_2(Y^*)} = u_2\left(F^{-1}_{Y^*}\right)$ follows from monotonicity of u_2, and the last inequality follows from the feasibility of f^* for Problem A.4. Hence, Y^* is feasible for Problem $\left(\widehat{\mathcal{P}}_{V_0}\right)$.

To show optimality of Y^* for Problem $\left(\widehat{\mathcal{P}}_{V_0}\right)$, let Y by any other feasible solution for Problem $\left(\widehat{\mathcal{P}}_{V_0}\right)$ and $f := F_Y^{-1}$ its quantile function. Then f is feasible for Problem A.4 by feasibility of Y for Problem $\left(\widehat{\mathcal{P}}_{V_0}\right)$, since

$$
V_0 \leqslant \int u_2(Y)\, dT_2 \circ P = \int_0^1 u_2\left(f(t)\right) T_2'(1-t)\, dt.
$$

Additionally, since $F_{-Y}(t) = -F_Y(1-t)$, for each $t \in \mathbb{R}$, a similar calculation as above yields, by monotonicity of u_1 that

$$
\begin{aligned}
\int u_1(-Y) dT_1 \circ P &= \int F^{-1}_{u_1(-Y)}(U)\, T_1'(1-U)\, dP = \int u_1\left(F^{-1}_{-Y}(U)\right) T_1'(1-U)\, dP \\
&= \int u_1\left(-F_Y^{-1}(1-U)\right) T_1'(1-U)\, dP = \int u_1\left(-F_Y^{-1}(U)\right) T_1'(U)\, dP \\
&= \int u_1\left(-f(U)\right) T_1'(U)\, dP = \int_0^1 u_1\left(-f(t)\right) T_1'(t)\, dt \\
&\leqslant \int_0^1 u_1\left(-f^*(t)\right) T_1'(t)\, dt = \int u_1(-Y^*)\, dT_1 \circ P,
\end{aligned}
$$

where the inequality follows from the optimality of f^* for Problem A.4. Therefore, Y^* is optimal for Problem $\left(\widehat{\mathcal{P}}_{V_0}\right)$. □

Now, for each $f \in \mathcal{Q}^*$, letting $v(t) = T_1^{-1}(t)$ and $z = v^{-1}(t) = T_1(t)$, for each $t \in [0,1]$ yields

$$\int_0^1 u_2\big(f(t)\big)\, T_2'(1-t)\, dt = \int_0^1 u_2\Big(f\big(v(z)\big)\Big)\, T_2'\big(1-v(z)\big)\, dv(z) = \int_0^1 u_2\big(q(t)\big)\, \Psi'(t)\, dt,$$

where $q = f \circ v$ and $\Psi(t) := 1 - T_2\big(1-v(t)\big) = \widetilde{T}_2\big(v(t)\big)$, for each t, where $\widetilde{T}_2 : [0,1] \to [0,1]$ is the conjugate of the function T_2, given by $\widetilde{T}_2(t) = 1 - T_2(1-t)$, so that $\widetilde{T}_2'(t) := T_2'(1-t)$, for all $t \in [0,1]$.

Now, consider the following problem.

Problem A.6.

$$\sup_{q \in \mathcal{Q}^*} \left\{ \int_0^1 u_1\big(-q(t)\big)\, dt : \int_0^1 u_2\big(q(t)\big)\, \Psi'(t)\, dt \geqslant V_0 \right\}.$$

Lemma A.7. *If q^* is optimal for Problem A.6, then $f^* := q^* \circ T_1$ is optimal for Problem A.4.*

Proof. Let q^* be optimal for Problem A.6, and $f^* := q^* \circ T_1$. Then $f^* \in \mathcal{Q}^*$ and $q^* = f^* \circ v$, where $v = T_1^{-1}$. Letting $z = T_1(t)$ yields

$$\int_0^1 u_2\big(f^*(t)\big)\, T_2'(1-t)\, dt = \int_0^1 u_2\Big(q^*\big(T_1(t)\big)\Big)\, T_2'(1-t)\, dt = \int_0^1 u_2\big(q^*(z)\big)\, \Psi'(z)\, dz \geqslant V_0,$$

where the inequality follows from the feasibility of q^* for Problem A.6. Hence f^* is feasible for Problem A.4.

To show optimality of f^* for Problem A.4, let f be feasible for Problem A.4 and $q := f \circ v$. Then $q \in \mathcal{Q}^*$ and

$$V_0 \leqslant \int_0^1 u_2\big(f(t)\big)\, T_2'(1-t)\, dt = \int_0^1 u_2\big(q(t)\big)\, \Psi'(t)\, dt.$$

Hence, q is feasible for Problem A.6. Therefore, by optimality of q^* is optimal for Problem A.6, it follows that

$$\begin{aligned}\int_0^1 u_1\big(-f^*(t)\big)\, T_1'(t)\, dt &= \int_0^1 u_1\big(-f^*(t)\big)\, dT_1(t) = \int_0^1 u_1\Big(-q^*\big(T_1(t)\big)\Big)\, dT_1(t) = \int_0^1 u_1\big(-q^*(z)\big)\, dz \\ &\geqslant \int_0^1 u_1\big(-q(z)\big)\, dz = \int_0^1 u_1\big(-f(t)\big)\, T_1'(t)\, dt.\end{aligned}$$

Hence, f^* is optimal for Problem A.4. □

In light of Lemma A.7, we turn our attention to solving Problem A.6. In order to do that, we will use a similar methodology to the one used by Xu [27], but adapted to the present setting. First, we recall the following result, due to He et al. [15, Appendix A].

Lemma A.8 (He et al. [15]). *Let f be a continuous real-valued function on a non-empty convex subset of $\mathbb{R}$ containing the interval $[0,1]$, and let g be its convex envelope on the interval $[0,1]$. Then,*

(1) *g is continuous and convex on $[0,1]$;*
(2) *$g(0) = f(0)$ and $g(1) = f(1)$;*
(3) *for all $x \in [0,1]$, $g(x) \leqslant f(x)$;*
(4) *g is affine on $\{x \in [0,1] : g(x) < f(x)\}$.*

Moreover,

(5) *if f is increasing, then so is g;*
(6) *if f is continuously differentiable on $(0,1)$, then g is continuously differentiable on $(0,1)$.*

Lemma A.9. *Let δ be the convex envelope of Ψ on $[0,1]$. Then for any $q \in \mathcal{Q}^*$,*

$$\int_0^1 u_2\big(q(t)\big)\,\Psi'(t)\,dt \leqslant \int_0^1 u_2\big(q(t)\big)\,\delta'(t)\,dt.$$

Proof. Let δ be the convex envelope of the function Ψ on $[0,1]$. Since $\delta(t) \leqslant \Psi(t)$, for all $t \in [0,1]$, $\Psi(0) = \delta(0)$, and $\Psi(1) = \delta(1)$, it follows from Fubini's Theorem that

$$\begin{aligned}
0 &\geqslant \int_0^1 \Big[\big(\Psi(1) - \delta(1)\big) - \big(\Psi(y) - \delta(y)\big)\Big]\,du_2\big(q(y)\big) \\
&= \int_0^1 \int_y^1 \big[\Psi'(x) - \delta'(x)\big]\,dx\;du_2\big(q(y)\big) \\
&= \int_0^1 \int_0^x \big[\Psi'(x) - \delta'(x)\big]\,du_2\big(q(y)\big)\;\,dx = \int_0^1 \left[\int_0^x du_2\big(q(y)\big)\right]\big[\Psi'(x) - \delta'(x)\big]\,dx \\
&= \int_0^1 \Big(u_2\big(q(t)\big) - u_2\big(q(0)\big)\Big)\big[\Psi'(t) - \delta'(t)\big]\,dt \\
&= \int_0^1 u_2\big(q(t)\big)\big[\Psi'(t) - \delta'(t)\big]\,dt - u_2\big(q(0)\big)\int_0^1 \big[\Psi'(t) - \delta'(t)\big]\,dt \\
&= \int_0^1 u_2\big(q(t)\big)\big[\Psi'(t) - \delta'(t)\big]\,dt - u_2\big(q(0)\big)\big(\Psi(1) - \delta(1) - (\Psi(0) - \delta(0)\big) \\
&= \int_0^1 u_2\big(q(t)\big)\big[\Psi'(t) - \delta'(t)\big]\,dt.
\end{aligned}$$

□

Now, consider the following problem.

Problem A.10.

$$\sup_{q\in\mathcal{Q}^*}\left\{\int_0^1 u_1\left(-q\left(t\right)\right)dt : \int_0^1 u_2\left(q\left(t\right)\right)\delta'\left(t\right)dt \geqslant V_0\right\}.$$

We first solve Problem A.10 and then show that the solution is also optimal for Problem A.6.

Lemma A.11. *If $q^* \in \mathcal{Q}^*$ satisfies*

(1) $\int_0^1 \delta'\left(t\right)u_2\left(q^*\left(t\right)\right)dt = V_0$;

(2) *there exists some $\lambda > 0$ such that for all $t \in (0,1)$,*

$$q^*\left(t\right) = \arg\max_y\left\{u_1\left(-y\right) + \lambda u_2\left(y\right)\delta'\left(t\right)\right\},$$

then q^* is optimal for Problem A.10.

Proof. Let $q^* \in \mathcal{Q}^*$ be such that the two conditions above are satisfied. Then q^* is feasible for Problem A.10. To show optimality, let $q \in \mathcal{Q}^*$ be any feasible solution for Problem A.10. Then, by definition of q^*, it follows that for each $t \in (0,1)$,

$$u_1\left(-q^*\left(t\right)\right) - u_1\left(-q\left(t\right)\right) \geqslant \lambda\left[\delta'\left(t\right)u_2\left(q\left(t\right)\right) - \delta'\left(t\right)u_2\left(q^*\left(t\right)\right)\right].$$

Hence, $\int_0^1 u_1\left(-q^*\left(t\right)\right)dt - \int_0^1 u_1\left(-q\left(t\right)\right)dt \geqslant \lambda\left[\int_0^1 \delta'\left(t\right)u_2\left(q\left(t\right)\right)dt - V_0\right] \geqslant 0$. Consequently, it follows that $\int_0^1 u_1\left(-q^*\left(t\right)\right)dt \geqslant \int_0^1 u_1\left(-q\left(t\right)\right)dt$, and so q^* is optimal for Problem A.10. ☐

Moreover, we obtain the following result.

Lemma A.12. *For each $\lambda > 0$, define the function q^*_λ by*

$$q^*_\lambda\left(t\right) := m^{-1}\left(\lambda\delta'\left(t\right)\right), \text{ for all } t \in (0,1), \tag{A.2}$$

where the function m is defined by

$$m\left(x\right) := \frac{u_1'\left(-x\right)}{u_2'\left(x\right)}.$$

Then,

(1) *for each $\lambda > 0$, $q^*_\lambda \in \mathcal{Q}^*$;*

(2) *there exists a $\lambda^* > 0$ such that $\int_0^1 \delta'\left(t\right)u_2\left(q^*_{\lambda^*}\left(t\right)\right)dt = V_0$;*

(3) *for all $t \in (0,1)$ and all $\lambda > 0$, $q^*_\lambda\left(t\right) = \arg\max_y\left\{u_1\left(-y\right) + \lambda u_2\left(y\right)\delta'\left(t\right)\right\}$.*

Proof. Assumptions 2.2 implies that u_1 and u_2 are increasing and continuously differentiable, and that u_1' and u_2' are decreasing. Therefore, the function m is continuously differentiable and

$$m'(x) = \frac{-u_1''(-x)\, u_2'(x) - u_1'(-x)\, u_2''(x)}{\left(u_2'(x)\right)^2} > 0.$$

Hence, the function m is continuous and increasing. Consequently, it is invertible, and its inverse is also increasing, by the Inverse Function Theorem. Thus, for each $\lambda > 0$, the convexity and continuity of δ imply that the function $g_\lambda^* = m^{-1}(\lambda\delta')$ is continuous and increasing. Therefore, for each $\lambda > 0$, $q_\lambda^* \in \mathcal{Q}^*$.

Now, for each $t \in (0,1)$ and $\lambda > 0$, the concavity of u_1 and u_2 yield the concavity of the function $y \mapsto \mathcal{M}_{\lambda,t}(y) := u_1(-y) + \lambda u_2(y)\,\delta'(t)$, since $\delta'(t) \geqslant 0$. Therefore, first-order conditions yield a global maximum of the function $\mathcal{M}_{\lambda,t}$ at $y^* = m^{-1}\left(\lambda\delta'(t)\right)$.

Finally, the existence of $\lambda^* > 0$ such that $\int_0^1 \delta'(t)\, u_2\left(q_{\lambda^*}^*(t)\right) dt = V_0$ follows from the monotonicity and continuity properties of δ' (see Lemma A.8), and from the Intermediate Value Theorem. □

Therefore, Lemmata A.9, A.11, and A.12 imply that for any $\lambda > 0$ and any $q \in \mathcal{Q}^*$,

$$\begin{aligned}
\int_0^1 \Big[u_1\Big(-q(t)\Big) + \lambda u_2\left(q(t)\right)\Psi'(t)\Big]dt &= \int_0^1 u_1\Big(-q(t)\Big)dt + \lambda\int_0^1 u_2\left(q(t)\right)\Psi'(t)\,dt \\
&\leqslant \int_0^1 u_1\Big(-q(t)\Big)dt + \lambda\int_0^1 u_2\left(q(t)\right)\delta'(t)\,dt = \int_0^1 \Big[u_1\Big(-q(t)\Big) + \lambda u_2\left(q(t)\right)\delta'(t)\Big]dt \\
&\leqslant \int_0^1 \Big[u_1\Big(-q_\lambda^*(t)\Big) + \lambda u_2\left(q_\lambda^*(t)\right)\delta'(t)\Big]dt,
\end{aligned}$$

where q_λ^* is as in eq. (A.2). Now, for all $\lambda > 0$, we have $q_\lambda^* \in \mathcal{Q}^*$ by Lemma A.12, and

$$dq_\lambda^*(t) = \lambda\left(m^{-1}\right)'\left(\lambda\delta'(t)\right)d\delta'(t)\,. \tag{A.3}$$

Letting $\mathcal{D} := \Big\{t \in [0,1] : \delta(t) \neq \Psi(t)\Big\} = \Big\{t \in [0,1] : \delta(t) < \Psi(t)\Big\}$, it follows that for any $\lambda > 0$,

$$\int_0^1 \left[\Psi(t) - \delta(t)\right] du_2\left(q_\lambda^*(t)\right) = \int_{\mathcal{D}} \left[\Psi(t) - \delta(t)\right] du_2\left(q_\lambda^*(t)\right).$$

But, since δ is affine on $\mathcal{D}$, $d\delta' = 0$ on $\mathcal{D}$, and it follows from eq. (A.3) that $dq_\lambda^*(t) = 0$ on $\mathcal{D}$, for all $\lambda > 0$. Consequently,

$$\int_0^1 \left[\Psi(t) - \delta(t)\right] du_2\left(q_\lambda^*(t)\right) = 0.$$

Therefore, applying Fubini's Theorem yields

$$0 = \int_0^1 \left[\Psi(t) - \delta(t)\right] du_2\left(q_\lambda^*(t)\right) = \int_0^1 u_2\left(q_\lambda^*(x)\right)\left[\Psi'(x) - \delta'(x)\right] dx.$$

Consequently, $\int_0^1 u_2\left(q_\lambda^*(t)\right)\Psi'(t)\,dt = \int_0^1 u_2\left(q_\lambda^*(t)\right)\delta'(t)\,dt$. Therefore, for all $\lambda > 0$ and all $q \in \mathcal{Q}^*$,

$$\begin{aligned}\int_0^1 \left[u_1\Big(-q(t)\Big) + \lambda u_2\left(q(t)\right)\Psi'(t)\right] dt &\leqslant \int_0^1 \left[u_1\Big(-q_\lambda^*(t)\Big) + \lambda u_2\left(q_\lambda^*(t)\right)\delta'(t)\right] dt \\ &= \int_0^1 \left[u_1\Big(-q_\lambda^*(t)\Big) + \lambda u_2\left(q_\lambda^*(t)\right)\Psi'(t)\right] dt.\end{aligned}$$

Hence, if λ^* is chosen such that $\int_0^1 u_2\left(q_{\lambda^*}^*(t)\right)\Psi'(t)\,dt = V_0$, then the optimal solution to Problem A.10 is given by $q_{\lambda^*}^*$. Thus, by Lemmata A.5, A.7, A.11, and A.12, the function

$$Y^* = q_{\lambda^*}^*\left(T_1(U)\right)$$

is optimal for Problem $\left(\widehat{\mathcal{P}}_{V_0}\right)$, where the function q_λ^* is given by (A.2).

This concludes the proof of Theorem A.3. □

A.3. **Uniqueness of Solutions to the Auxiliary Problem.**

Lemma A.13. *For a given $V_0 \in \mathbb{R}$, let Y_1 be optimal for Problem $\left(\widehat{\mathcal{P}}_{V_0}\right)$ and Y_2 be feasible for Problem $\left(\widehat{\mathcal{P}}_{V_0}\right)$. Then Y_2 is also optimal for Problem $\left(\widehat{\mathcal{P}}_{V_0}\right)$ if and only if for a.e. $t \in \mathbb{R}$,*

$$P\left(\{s \in S : Y_1(s) > t\}\right) = P\left(\{s \in S : Y_2(s) > t\}\right).$$

Proof. We start with the "if" statement. First suppose that for a.e. $t \in \mathbb{R}$,

$$P\left(\{s \in S : Y_1(s) > t\}\right) = P\left(\{s \in S : Y_2(s) > t\}\right).$$

Then by definition of the Choquet integral, it follows that

$$\int u_1(-Y_1)\ dT_1 \circ P = \int u_1(-Y_2)\ dT_1 \circ P \quad \text{and} \quad \int u_2(Y_1)\ dT_2 \circ P = \int u_2(Y_2)\ dT_2 \circ P.$$

Therefore, Y_2 is also optimal for Problem $\left(\widehat{\mathcal{P}}_{V_0}\right)$.

We proceed with the "only if" statement. Problem $\left(\widehat{\mathcal{P}}_{V_0}\right)$ has a solution due to Theorem A.3. Suppose that there are two solutions Y_1, Y_2 to Problem $\left(\widehat{\mathcal{P}}_{V_0}\right)$ such that it does not hold that for a.e. $t \in \mathbb{R}$, $P\left(\{s \in S : Y_1(s) > t\}\right) = P\left(\{s \in S : Y_2(s) > t\}\right)$. Since the probability space

(S, Σ, P) is non-atomic, there exists a $Y_2^c \in B(\Sigma)$ such that Y_2^c, Y_2 are identically distributed and Y_2^c is comonotonic with Y_1. Thus,

$$\int u_1\left(-Y_1\right)\ dT_1 \circ P = \int u_1\left(-Y_2\right)\ dT_1 \circ P = \int u_1\left(-Y_2^c\right)\ dT_1 \circ P. \tag{A.4}$$

Let $\widetilde{\Sigma}$ be the σ-algebra on S generated by the random variable $\frac{1}{2}(Y_1 + Y_2^c)$. Define the probability measure Q_1 on $\left(S, \widetilde{\Sigma}\right)$ by $Q_1(-\frac{1}{2}(Y_1+Y_2^c) > t) := T_1(P(-\frac{1}{2}(Y_1+Y_2^c) > t))$, for $t \in \mathbb{R}$. Since the σ-algebra is generated by $\frac{1}{2}(Y_1 + Y_2^c)$ and since T_1 is increasing and continuous, Q_1 is indeed a probability measure on $\left(S, \widetilde{\Sigma}\right)$. The three random variables $-Y_1, -Y_2^c, -\frac{1}{2}(Y_1 + Y_2^c)$ are all $\widetilde{\Sigma}$-measurable and comonotonic with $-\frac{1}{2}(Y_1+Y_2^c)$. Therefore, for $X = -Y_1, -Y_2^c, -\frac{1}{2}(Y_1+Y_2^c)$, it holds that

$$\begin{aligned}\int u_1\left(X\right)\ dT_1 \circ P &= \int_{-\infty}^{0} \big(T_1 \circ P(u_1(X) > z) - 1\big)\, dz + \int_0^\infty T_1 \circ P(u_1(X) > z) dz \\ &= \int_{-\infty}^{0} \big(Q_1(u_1(X) > z) - 1\big)\, dz + \int_0^\infty Q_1(u_1(X) > z) dz = E^{Q_1}[u_1(X)] = \int_S u_1(X(s)) dQ_1(s),\end{aligned}$$

where E^{Q_1} is the expectation under the probability measure Q_1. Note that this relies on the useful fact that $-\frac{1}{2}(Y_1 + Y_2^c)$ and X are comonotonic. Then, since it does not hold that Y_1 and Y_2^c are equal in distribution, it follows that $P(-Y_1 \neq -Y_2^c) > 0$. Define $\mathcal{A} \in \widetilde{\Sigma}$ such that $-Y_1(s) \neq -Y_2^c(s)$ for all $s \in \mathcal{A}$ and $-Y_1(s) = -Y_2^c(s)$ for all $s \in S \backslash \mathcal{A}$. Since the function T_1 is strictly increasing, it holds for $t \in \mathbb{R}$ that $Q_1(-\frac{1}{2}(Y_1 + Y_2^c) > t)$ strictly decreases if and only if $P(-\frac{1}{2}(Y_1 + Y_2^c) > t)$ strictly decreases, and thus Q_1 is an equivalent probability measure to P on $\left(S, \widetilde{\Sigma}\right)$. It hence follows that $Q_1(\mathcal{A}) := \int_{\mathcal{A}} dQ_1(s) > 0$. By eq. (A.4), it follows that

$$\int_{\mathcal{A}} u_1\left(-Y_1(s)\right) dQ_1(s) = \int_{\mathcal{A}} u_1\left(-Y_2^c(s)\right) dQ_1(s).$$

Recall that $\int u_1\left(-Y_1\right)\ dT_1 \circ P = E^{Q_1}[u_1\left(-Y_1\right)]$. Moreover,

$$\begin{aligned}&\int u_1\left(-\left(\frac{1}{2}Y_1 + \frac{1}{2}Y_2^c\right)\right)\ dT_1 \circ P = E^{Q_1}\left[u_1\left(-\left(\frac{1}{2}Y_1 + \frac{1}{2}Y_2^c\right)\right)\right] \\ &= \int_S u_1\left(-\left(\frac{1}{2}Y_1(s) + \frac{1}{2}Y_2^c(s)\right)\right) dQ_1 \\ &= \int_{\mathcal{A}} u_1\left(-\left(\frac{1}{2}Y_1(s) + \frac{1}{2}Y_2^c(s)\right)\right) dQ_1 + \int_{S\backslash\mathcal{A}} u_1\left(-\left(\frac{1}{2}Y_1(s) + \frac{1}{2}Y_2^c(s)\right)\right) dQ_1 \\ &= \int_{\mathcal{A}} u_1\left(-\left(\frac{1}{2}Y_1(s) + \frac{1}{2}Y_2^c(s)\right)\right) dQ_1 + \int_{S\backslash\mathcal{A}} u_1\left(-Y_1(s)\right) dQ_1 \\ &> \int_{\mathcal{A}} \left(\frac{1}{2}u_1\left(-Y_1(s)\right) + \frac{1}{2}u_1\left(-Y_2^c(s)\right)\right) dQ_1 + \int_{S\backslash\mathcal{A}} u_1\left(-Y_1(s)\right) dQ_1\end{aligned}$$

$$= \frac{1}{2}\int_{\mathcal{A}} u_1\left(-Y_1(s)\right) dQ_1 + \frac{1}{2}\int_{\mathcal{A}} u_1\left(-Y_2^c(s)\right) dQ_1 + \int_{S\setminus\mathcal{A}} u_1\left(-Y_1(s)\right) dQ_1$$
$$= \int_{\mathcal{A}} u_1\left(-Y_1(s)\right) dQ_1 + \int_{S\setminus\mathcal{A}} u_1\left(-Y_1(s)\right) dQ_1$$
$$= \int_{S} u_1\left(-Y_1(s)\right) dQ_1 = E^{Q_1}[u_1\left(-Y_1\right)] = \int u_1\left(-Y_1\right)\ dT_1 \circ P,$$

where the strict inequality follows from $u_1(\frac{1}{2}x + \frac{1}{2}y) > \frac{1}{2}u_1(x) + \frac{1}{2}u_1(y)$ whenever $x \neq y$, which follows from strict concavity of u_1. Via similar concavity arguments, we also find

$$\int u_2\left(\frac{1}{2}Y_1 + \frac{1}{2}Y_2^c\right)\ dT_2 \circ P \geqslant \int u_2\left(Y_1\right) dT_2 \circ P \geqslant V_0.$$

Thus, $\frac{1}{2}Y_1 + \frac{1}{2}Y_2^c$ strictly improves Y_1, and so Y_1 is not a solution of Problem $\left(\widehat{\mathcal{P}}_{V_0}\right)$, a contradiction. □

A.4. **Proof of Theorem 3.2.** Theorem 3.2 is a direct consequence of Lemma A.2, Theorem A.3, and Lemma A.13. Let

$$Y^* = m^{-1}\left(\lambda^*\delta'\Big(T_1\left(U\right)\Big)\right),$$

where $\lambda^* > 0$, and where U, m, and δ are as defined in Theorem A.3. Then by Theorem A.3, Y^* is optimal for Problem $\left(\widehat{\mathcal{P}}_{V_0}\right)$, with $V_0 := \int u_2\left(Y^*\right) dT_2 \circ P$. Consequently, by Lemma A.2$-(ii)$, Y^* is Pareto optimal.

Conversely, suppose that $Y^* \in B\left(\Sigma\right)$ is Pareto optimal. Then by Lemma A.2$-(i)$, Y^* solves Problem $\left(\widehat{\mathcal{P}}_{V_0}\right)$ with $V_0 := \int u_2\left(Y^*\right)\ dT_2 \circ P$. By Theorem A.3, a solution is given by

$$\hat{Y} = m^{-1}\left(\lambda^*\delta'\Big(T_1\left(U\right)\Big)\right),$$

where $\lambda^* > 0$ is chosen such that $\int u_2\left(\hat{Y}\right) dT_2 \circ P = V_0$, and where U, m, and δ are as defined in Theorem A.3. By Lemma A.13, all solutions to Problem $\left(\widehat{\mathcal{P}}_{V_0}\right)$ have the same distribution as $\hat{Y}$. Hence, for every Pareto optimal risk-sharing contract $Y^* \in B\left(\Sigma\right)$, there exists a $\lambda^* > 0$ such that the risk-sharing contract Y^* has the same distribution as $m^{-1}\left(\lambda^*\delta'\Big(T_1\left(U\right)\Big)\right)$.

□

A.5. **A Remark.** We could have characterized the set of Pareto optimal risk-sharing contracts using a different parametrization. Indeed, consider the following auxiliary problem:

Problem A.14 $(\overline{\mathcal{P}}_{\Gamma_0})$**.** *For* $\Gamma_0 \in \mathbb{R}$,

$$\left(\overline{\mathcal{P}}_{\Gamma_0}\right) \qquad \sup_{Y \in B(\Sigma)} \left\{ \int u_2\left(Y\right)\ dT_2 \circ P : \int u_1\left(-Y\right)\ dT_1 \circ P \geqslant \Gamma_0 \right\}.$$

By a proof similar to that of Lemma A.2, we obtain the following result.

Lemma A.15.

(i) *If the risk-sharing contract* $Y^* \in B\left(\Sigma\right)$ *is Pareto optimal, then it solves Problem* $\left(\overline{\mathcal{P}}_{\Gamma_0}\right)$ *with* $\Gamma_0 := \int u_1\left(-Y^*\right)\ dT_1 \circ P$*;*

(ii) *for a given* $\Gamma_0 \in \mathbb{R}$*, any solution to Problem* $\left(\overline{\mathcal{P}}_{\Gamma_0}\right)$ *is Pareto optimal;*

(iii) *if* $Y^* \in B\left(\Sigma\right)$ *solves Problem* $\left(\overline{\mathcal{P}}_{\Gamma_0}\right)$ *for a given* $\Gamma_0 \in \mathbb{R}$*, then* $\int u_1\left(-Y^*\right)\ dT_1 \circ P = \Gamma_0$.

By a proof similar to that of Theorem A.3, we also obtain the following result.

Theorem A.16. *For a given* $\Gamma_0 \in \mathbb{R}$*, the risk-sharing contract*

$$\overline{Y}^* := \overline{m}^{\ -1}\left(\kappa^* \overline{\delta}'\left(\widetilde{T}_2\left(U\right)\right)\right)$$

is optimal for Problem $\left(\overline{\mathcal{P}}_{\Gamma_0}\right)$*, where:*

- U *is a random variable on* (S, Σ, P) *with a uniform distribution on* $(0,1)$*;*
- $\overline{m}\left(x\right) := \frac{u_2'(x)}{u_1'(-x)}$*, for all* $x \geqslant 0$*;*
- $\overline{\delta}$ *is the concave envelope on* $[0,1]$ *of* $\overline{\Psi}\left(t\right) := T_1\left(\widetilde{T}_2^{-1}\left(t\right)\right)$*;*
- $\widetilde{T}_2\left(t\right) = 1 - T_2\left(1-t\right)$*; and,*
- $\kappa^* > 0$ *is chosen such that* $\int u_1\left(-Y^*\right) dT_1 \circ P = \Gamma_0$.

Likewise, by a proof similar to that of Lemma A.13, we also obtain the following result.

Lemma A.17. *For a given* $\Gamma_0 \in \mathbb{R}$*, let* $\overline{Y}_1$ *be optimal for Problem* $\left(\overline{\mathcal{P}}_{\Gamma_0}\right)$ *and* $\overline{Y}_2$ *be feasible for Problem* $\left(\overline{\mathcal{P}}_{\Gamma_0}\right)$*. Then* $\overline{Y}_2$ *is also optimal for Problem* $\left(\overline{\mathcal{P}}_{\Gamma_0}\right)$ *if and only if for a.e.* $t \in \mathbb{R}$*,*

$$P\left(\{s \in S : \overline{Y}_1(s) > t\}\right) = P\left(\{s \in S : \overline{Y}_2(s) > t\}\right).$$

Consequently, Lemma A.15, Theorem A.16 and Lemma A.17 provide an alternative way of characterizing the set of Pareto optima.

Appendix B. Proof of Theorem 3.3

Let the functions Ψ and δ be as in Theorem A.3. For $i \in \{1,2\}$, let $\widetilde{T}_i$ be the conjugate of T_i, defined by $\widetilde{T}_i(z) := 1 - T_i(1-z)$, for each $z \in [0,1]$. Then letting $x = 1-z$, it follows that for each $x \in [0,1]$ and for $i \in \{1,2\}$, $T_i(x) = 1 - \widetilde{T}_i(1-x)$. Therefore, for $i \in \{1,2\}$, and for each $x, z \in [0,1]$,

$$\widetilde{T}_i'(z) = T_i'(1-z) \quad \text{and} \quad T_i'(x) = \widetilde{T}_i'(1-x)\,;$$
$$\widetilde{T}_i''(z) = -T_i''(1-z) \quad \text{and} \quad T_i''(x) = -\widetilde{T}_i''(1-x)\,.$$

Suppose that there exits $i \in \{1,2\}$ such that for all $z \in (0,1)$,

$$(\star) \qquad \frac{\widetilde{T}_i''(z)}{\widetilde{T}_i'(z)} > \frac{T_j''(z)}{T_j'(z)},$$

where $j = 3-i$. Then, for all $z = 1-x \in (0,1)$,

$$\frac{-T_i''(x)}{T_i'(x)} = \frac{-T_i''(1-z)}{T_i'(1-z)} = \frac{\widetilde{T}_i''(z)}{\widetilde{T}_i'(z)} > \frac{T_j''(z)}{T_j'(z)} = \frac{-\widetilde{T}_j''(1-z)}{\widetilde{T}_j'(1-z)} = \frac{-\widetilde{T}_j''(x)}{\widetilde{T}_j'(x)},$$

and so, for each $x \in (0,1)$,

$$\frac{\widetilde{T}_j''(x)}{\widetilde{T}_j'(x)} > \frac{T_i''(x)}{T_i'(x)}.$$

Hence, without loss of generality, we can assume that $i = 1$ and $j = 2$, so that for each $x \in (0,1)$,

$$\frac{\widetilde{T}_2''(x)}{\widetilde{T}_2'(x)} > \frac{T_1''(x)}{T_1'(x)}.$$

This then implies that the function Ψ is convex, and hence $\delta \equiv \Psi$. The rest then follows from Theorem 3.2. □

Appendix C. Proof of Theorem 4.1

First, we show that (1) is equivalent to (2), i.e., $\Psi(t) \geqslant t$ for all $t \in [0,1]$ if and only if there exists a Pareto optimal no-betting allocation. By virtue of Theorem 3.2, there exists a Pareto optimal no-betting allocation if and only if $\delta(t) = t$ for all $t \in [0,1]$. Thus, it is sufficient to show $\delta(t) = t$ for all $t \in [0,1]$ if and only $\Psi(t) \geqslant t$ for all $t \in [0,1]$. If $\delta(t) = t$ for all $t \in [0,1]$, then it follows directly from Lemma A.8(3) that $\Psi(t) \geqslant t$. Let $\Psi(t) \geqslant t$. By Lemma A.8(2) it holds that $\delta(0) = \Psi(0) = 0$ and $\delta(1) = \Psi(1) = 1$. Since δ is convex, it holds that $\delta(t) \leqslant t$. The largest convex function is thus $\delta(t) = t$, which satisfies $\delta(t) \leqslant \Psi(t)$. Thus, $\delta(t) = t$ is the convex envelope.

Next, we show that (2) is equivalent with (3). The relation that (3) implies (2) is trivial. We next show that (2) implies (3), i.e., existence of a Pareto optimal no-betting allocation implies that any Pareto optimal risk-sharing contract is a no-betting allocation. If there exists a Pareto optimal no-betting allocation, this implies by Theorem 3.2 that there exists some $\lambda^* > 0$ such that $m^{-1}\left(\lambda^*\delta'\Big(T_1(U)\Big)\right)$ has the same distribution as a deterministic random variable. The function m is strictly increasing, and thus m^{-1} is strictly increasing. Moreover, T_1 is assumed to be strictly increasing. Thus, it holds that there exists a $c > 0$ such that $\delta'(t) = c$ for all $t \in [0,1]$ almost everywhere. Since $\delta(0) = 0$ and $\delta(1) = 1$ by Lemma A.8, this implies $\delta(t) = t$. Hence, for any $\lambda > 0$, $m^{-1}\left(\lambda\delta'\Big(T_1(U)\Big)\right)$ is deterministic. Consequently, by Theorem 3.2, any Pareto optimal risk-sharing contract is a no-betting allocation.

Next, we show that (2) is equivalent with (4). The relation that (4) implies (2) is trivial. We next show that (2) implies (4), i.e., we show that existence of a Pareto optimal no-betting allocation implies that every no-betting allocation is Pareto optimal. As argued above, if there exists a Pareto optimal no-betting allocation, then Theorem 3.2 implies that $\delta(t) = t$, for each $t \in [0,1]$. Consequently, $\delta' \equiv 1$. Now, suppose that $Y = c \in \mathbb{R}$ is a no-betting allocation. Letting $\lambda^* := m(c)$, where m is defined in Theorem 3.2, it follows that

$$Y = c = m^{-1}(\lambda^*) = m^{-1}\left(\lambda^*\delta'\Big(T_1(U)\Big)\right),$$

since $\delta' \equiv 1$. Consequently, it follows again from Theorem 3.2 that Y is Pareto optimal. □

Appendix D. Proof of Proposition 4.2

Let the functions Ψ and δ be as in Theorem A.3, and let the functions $\overline{\Psi}$ and $\overline{\delta}$ be as in Theorem A.16. For $i \in \{1,2\}$, let $\widetilde{T}_i$ be the conjugate of T_i, defined by $\widetilde{T}_i(z) := 1 - T_i(1-z)$, for each $z \in [0,1]$. Then letting $x = 1 - z$, it follows that for each $x \in [0,1]$ and for $i \in \{1,2\}$, $T_i(x) = 1 - \widetilde{T}_i(1-x)$. Therefore, for $i \in \{1,2\}$, and for each $x, z \in (0,1)$,

$$\widetilde{T}_i'(z) = T_i'(1-z) \quad \text{and} \quad T_i'(x) = \widetilde{T}_i'(1-x);$$
$$\widetilde{T}_i''(z) = -T_i''(1-z) \quad \text{and} \quad T_i''(x) = -\widetilde{T}_i''(1-x).$$

Suppose that there exits $i \in \{1,2\}$ such that for all $z \in (0,1)$,

$$(\star\star) \qquad \frac{\widetilde{T}_i''(z)}{\widetilde{T}_i'(z)} \leqslant \frac{T_j''(z)}{T_j'(z)},$$

where $j = 3 - i$. Then, for all $z = 1 - x \in (0,1)$,

$$\frac{-T_i''(x)}{T_i'(x)} = \frac{-T_i''(1-z)}{T_i'(1-z)} = \frac{\widetilde{T}_i''(z)}{\widetilde{T}_i'(z)} \leqslant \frac{T_j''(z)}{T_j'(z)} = \frac{-\widetilde{T}_j''(1-z)}{\widetilde{T}_j'(1-z)} = \frac{-\widetilde{T}_j''(x)}{\widetilde{T}_j'(x)},$$

and so, for each $x \in (0,1)$,

$$\frac{\widetilde{T}_j''(x)}{\widetilde{T}_j'(x)} \leqslant \frac{T_i''(x)}{T_i'(x)}.$$

Hence, without loss of generality, we can assume that $i = 1$ and $j = 2$, so that for each $x \in (0,1)$,

$$\frac{\widetilde{T}_2''(x)}{\widetilde{T}_2'(x)} \leqslant \frac{T_1''(x)}{T_1'(x)}.$$

This then implies that the function Ψ is concave and the function $\overline{\Psi}$ is convex. In turn, this implies that the functions δ and $\overline{\delta}$ are both linear, and that $\delta' \equiv \overline{\delta}' \equiv 1$. Consequently, the risk-sharing contracts Y^* and $\overline{Y}^*$ given in Theorems A.3 and A.16, respectively, are constants. That is, there exists a Pareto optimal no-betting allocation.

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